



Computation of k-out-of-n System Reliability *via* Reduced Ordered Binary Decision Diagrams

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Authors' contributions

This work was carried out in collaboration between both authors. Author AMAR designed the study, managed the analysis, wrote the first draft of the manuscript and managed literature survey. Author AMA contributed to the analysis, implemented the algorithm, drew the figures and contributed to literature survey. Both authors read and approved the final manuscript.

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Abstract

A prominent reliability model is that of the partially-redundant (k-out-of-n) system. We use algebraic as well as signal-flow-graph methods to explore and expose the AR algorithm for computing k-out-of-n reliability. We demonstrate that the AR algorithm is, in fact, both a recursive and an iterative implementation of the strategy of Reduced Ordered Binary Decision Diagrams (ROBDDs). The underlying ROBDD for the AR recursive algorithm is represented by a compact Signal Flow Graph (SFG) that is used to deduce AR iterative algorithms of quadratic temporal complexity and linear spatial complexity. Extensions of the AR algorithm for (single or scalar) threshold, double-threshold, vector-threshold, and k-to-l-out-of-n systems have similar ROBDD interpretations.

Keywords: AR algorithm; k-out-of-n system; reduced ordered binary decision diagram; reliability; signal flow graph.

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1 Introduction

The k-out-of-n:G (F) system, is “a system of n components that functions (fails) if at least k out of its n components function (fail)”. Situations in which this system serves as a useful model are frequently encountered in practice [1-4]. The k-out-of-n system plays a central role for the general class of coherent systems, as it can be used to express or approximate the reliability of such systems with knowledge of the signatures or the destruction spectrum of such systems [4-5].

This paper deals with the evaluation of k-out-of-n system reliability *via* the Reduced-Ordered-Binary-Decision-Diagram (ROBDD) strategy. The ROBDD strategy was proposed by Bryant [6] as an extension of the Binary-Decision-Diagram (BDD) methodology of Akers [7]. The ROBDD deals with *general* switching (two-valued Boolean) functions, and is now considered the state-of-the-art data structure for handling such functions. The ROBDD has many applications in reliability theory and engineering [1,8-14]. Most of the ROBDD applications in reliability concentrate on areas which have clear-cut Boolean interpretations, such as those of fault-tree analysis or event-tree analysis [9]. In the worst case, ROBDD applications have exponential (and hence, intractable) complexity. One of a few notable exceptions is the case of applying the ROBDD to k-out-of-n systems.

The purpose of this paper is to demonstrate that simple application of the ROBDD strategy to k-out-of-n system reliability analysis produces an already existing algorithm, namely the *recursive* AR algorithm of Rushdi [3,15-17]. Careful implementations of the ROBDD strategy can be used to recover *iterative* versions of the AR algorithm.

The organization of the remainder of this paper is as follows. Section 2 reviews the AR algorithm both in the Boolean and probability domains and illustrates it *via* a signal-flow-graph representation. Section 3 explains why the AR algorithm is a special case of the ROBDD strategy, while Sec. 4 demonstrates the findings of Sec. 3 visually from the perspective of k-out-of-5:G systems, where k ranges from 1 to 5. Section 5 concludes the paper.

2 The AR Algorithm

The success $S(\mathbf{X}_n)$ of a certain system in terms of the successes $\mathbf{X}_n = [X_1 \ X_2 \dots X_{j-1} \ X_j \ X_{j+1} \dots X_n]^T$ of its components can be expanded *via* the Boole-Shannon expansion *w.r.t.* the success X_j of component j [17]

$$S(\mathbf{X}_n) = \bar{X}_j (S(\mathbf{X}_n) / \bar{X}_j) \vee X_j (S(\mathbf{X}_n) / X_j), \quad 1 \leq j \leq n. \quad (1)$$

This expansion expresses $S(\mathbf{X}_n)$ in terms of two Boolean quotients (also called ratios, subfunctions or restrictions) [18-19]

$$S(\mathbf{X}_n) / \bar{X}_j = S(\mathbf{X}_n) \rfloor_{X_j=0}, \quad (2)$$

$$S(\mathbf{X}_n) / X_j = S(\mathbf{X}_n) \rfloor_{X_j=1}. \quad (3)$$

Rushdi [15,17] proved that for the success of a k-out-of-n:G system, denoted herein as $S(k, n, \mathbf{X}_n)$, the above quotients (for $1 \leq j \leq n$) are

$$S(k, n, \mathbf{X}_n) / \bar{X}_j = S(k, n-1, \mathbf{X}_n/X_j), \quad (4)$$

$$S(k, n, \mathbf{X}_n) / X_j = S(k-1, n-1, \mathbf{X}_n/X_j), \quad (5)$$

where $\mathbf{X}_n/\mathbf{X}_j = [X_1 \ X_2 \ \dots \ X_{j-1} \ X_{j+1} \ \dots \ X_n]^T$. Note that the slash (/) in (1) – (5) has the meaning of restriction (or with a little abuse of notation, that of division), while in $(\mathbf{X}_n/\mathbf{X}_j)$ its meaning resembles that of a set difference, and hence it is used to exclude an element X_j from an n -element vector \mathbf{X}_n to produce an $(n - 1)$ element vector $(\mathbf{X}_n/\mathbf{X}_j)$. When equations (1) – (5) are combined (with n set to j), one obtains

$$S(k, j, \mathbf{X}_j) = \bar{X}_j S(k, j-1, \mathbf{X}_{j-1}) \vee X_j S(k-1, j-1, \mathbf{X}_{j-1}), \quad 1 \leq k \leq j \leq n. \quad (6)$$

The above recursive relation should be used in conjunction with the boundary conditions

$$S(0, j, \mathbf{X}_j) = 1, \quad j \geq 0. \quad (7a)$$

$$S(j+1, j, \mathbf{X}_j) = 0, \quad j \geq 0. \quad (7b)$$

Since Equations (6) and (7) are probability-ready expressions (PREs) [19], they are converted immediately to the probability domain by replacing Boolean variables by their expectations and substituting the arithmetic operations of multiplication and addition for their logical counterparts of ANDing and ORing, namely

$$R(k, j, \mathbf{p}_j) = q_j R(k, j-1, \mathbf{p}_{j-1}) + p_j R(k-1, j-1, \mathbf{p}_{j-1}), \quad 1 \leq k \leq j \leq n, \quad (8)$$

$$R(0, j, \mathbf{p}_j) = 1.0, \quad j \geq 0, \quad (9a)$$

$$R(j+1, j, \mathbf{p}_j) = 0.0, \quad j \geq 0. \quad (9b)$$

Equations (6) and (7) are the basis of the AR algorithm in the Boolean domain, while equations (8) and (9) constitute its essence in the probability domain. This algorithm is a quadratic-time *iterative* algorithm that has the beautiful characteristic of having a common complexity of $O(k(n - k + 1))$ for computing both the unreliability and reliability of either the k -out-of- n :F system or the k -out-of- n :G system [3,19]. Figs. 1 and 2 are signal flow graphs (SFGs) that represent the underlying relations (6) and (7) for the AR algorithm in the Boolean domain, and the transformed relations (8) and (9) for that algorithm in the probability domain. The figures are drawn on a rectangular grid of a vertical coordinate $k_1 = k$ ($k_1 \geq 0$) and a horizontal coordinate $k_2 = j - k$ ($k_2 \geq -1$). Shaded circles represent nodes expressed recursively, while squares represent boundary conditions of specific values 0 and 1 for white and black nodes, respectively. Figs. 1 and 2 are useful for visualizing both the recursive and iterative versions of the AR algorithm [17]. At least three iterative versions are possible *via* the *recursion removal* attained when one traverses or sweeps the rows, columns, or secondary diagonals in Figs. 1 and 2. The arrows in Figs. 1 and 2 indicate signal-flow-graph relations, and as such, are opposite to what would be used conventionally in an expansion graph.

3 Comparison of the AR Algorithm with the ROBDD Strategy

The AR algorithm is, in fact, an implementation of the Reduced-Ordered-Binary-Decision-Diagram (ROBDD) strategy when this strategy is adapted for computing the k -out- n -reliability. The ROBDD deals with *general* switching (two-valued Boolean) functions. The AR algorithm, however, handles a class of switching functions that are both *monotonically non-decreasing* and *totally symmetric*. Apart from this restriction in applicability, the recursive version of AR algorithm has exactly the same features as the ROBDD algorithm, namely:

1. Both the AR and ROBDD algorithms are based on the Boole-Shannon expansion in the Boolean domain (1), and both use boundary conditions such as those in (7) to terminate this expansion.
2. Both algorithms visit the variables in a certain order, typically monotonically ascending or monotonically descending.
3. Both algorithms reduce the resulting expansion tree (which is exponential in size) to a rooted acyclic graph that is both canonical and hopefully compact or sub-exponential. The reduction rules [4] require (a) merging isomorphic subtrees, and (b) deletion of any useless intermediate node

whose two “outgoing” edges point both to the same child node. For the AR algorithm, the resulting graph is not only ‘hopefully’ compact, but is guaranteed to correspond to quadratic temporal complexity and linear spatial complexity. Moreover, in the AR algorithm, reduction rule (b) is never needed, as these are no useless intermediate nodes. In signal-flow-graph notation, two ‘outgoing’ edges are, in fact, two ‘incoming’ transmittances. Every circled node in Figs. 1 and 2 receives transmittances from exactly two distinct nodes that do not share the same value.

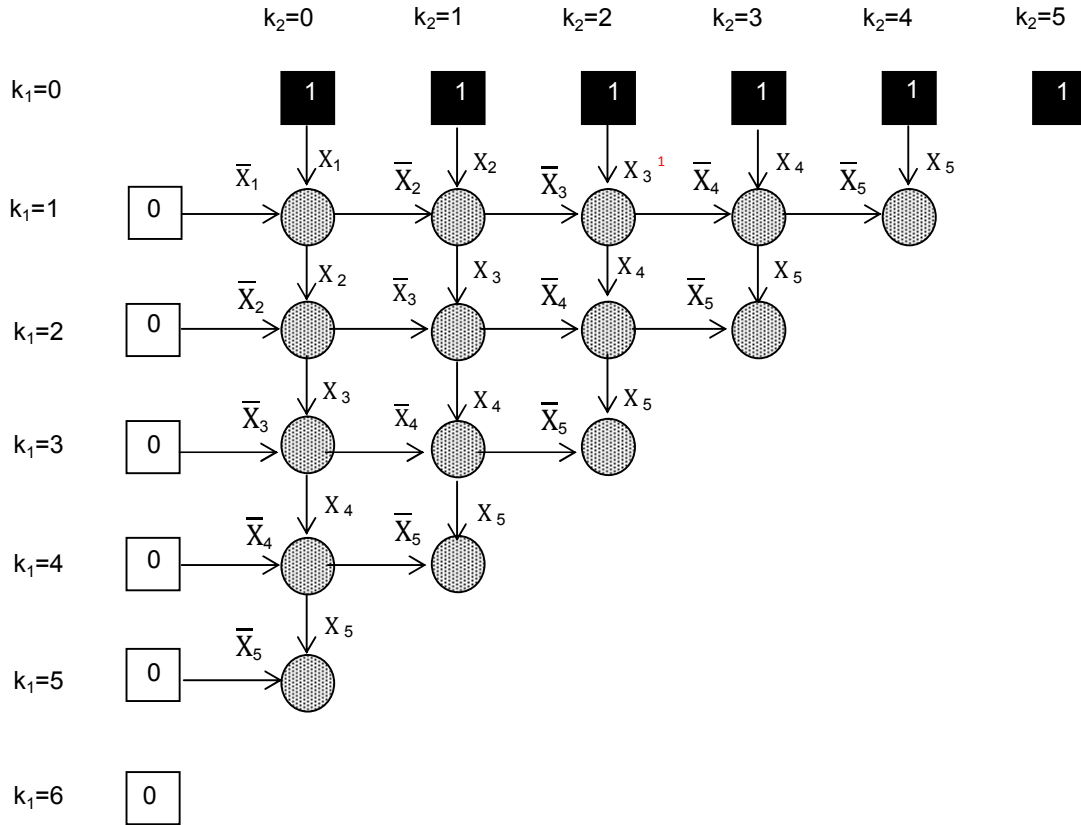


Fig. 1. The signal flow graph representing the underlying relations of the AR algorithm in the Boolean domain (operators of Boolean ANDing and ORing are assumed)

Therefore, the AR algorithm can be seen to be a subclass of ROBDD algorithms that is tailored specifically for the following equivalent purposes:

- (a) Handling monotone symmetric switching functions,
- (b) Computing k -out-of- n reliability; and
- (c) Computing the Complementary Cumulative Distribution Function (CCDF) of the generalized Binomial Distribution.

Contrary to claims made in the literature (see, *e.g.*, [20]), the AR algorithm is not only recursive but it has several iterative versions as well, that first appeared in [15] and reproduced in expository detail in [3,17]. There are ROBDD-like extensions of the AR algorithm, which handle (single or scalar) threshold systems [16,21], double-threshold systems [22], vector-threshold systems [23], and k -to- l -out-of- n systems [24].

4 Demonstration via the k-out-of-5: G System

To demonstrate that the AR algorithm is the ROBDD algorithm for computing the k-out-of-n reliability, we use Fig. 3 to depict the various ROBDDs used in the evaluation of $R(k, 5, \mathbf{p})$, $1 \leq k \leq 5$, which are each analogous to the related subgraphs in either Figs. 1 or 2. In an ROBDD, a node does not depict a physical entity such as a reliability value but represents a decision point and hence is labeled by the pertinent decision variable. The two emanating edges of each node are labeled by 0 and 1 values representing the two states of the decision variable. These 0 and 1 values are analogous to the \bar{X}_i and X_i transmittances on the SFG of Fig. 1, and also to the q_i and p_i transmittances on the SFG of Fig. 2.

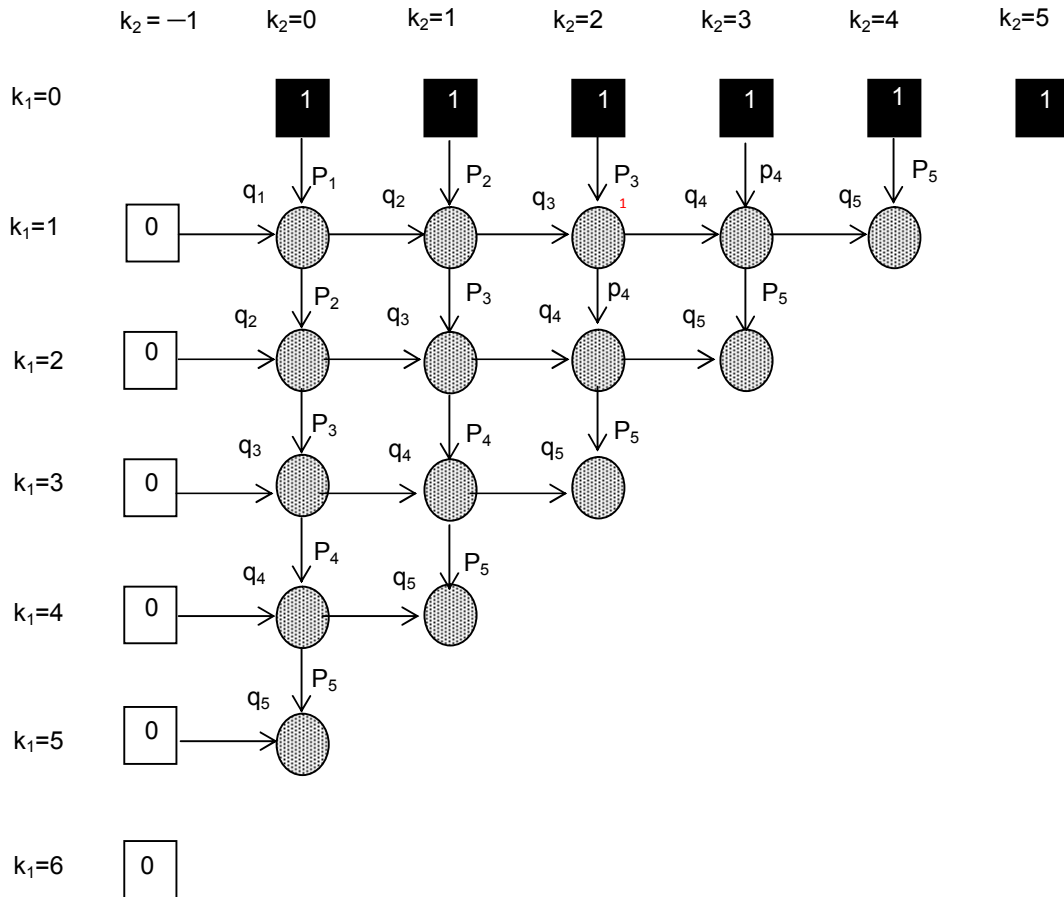
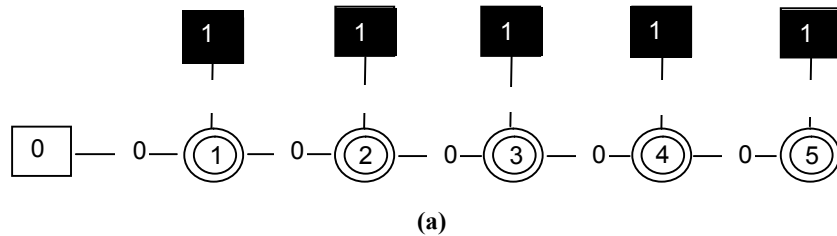
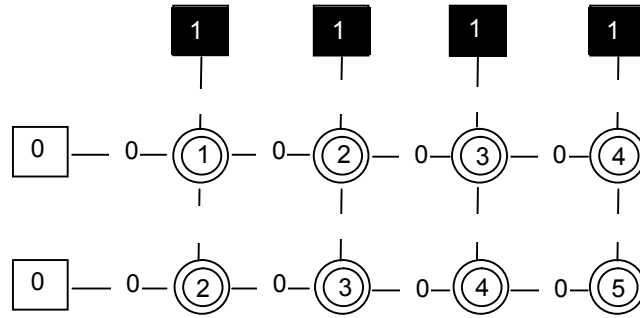
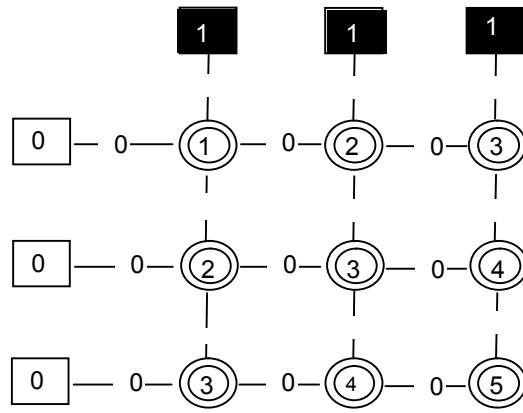


Fig. 2. The Signal Flow Graph representing the underlying relations of the AR algorithm in the probability domain (operators of arithmetic addition and multiplication are assumed)

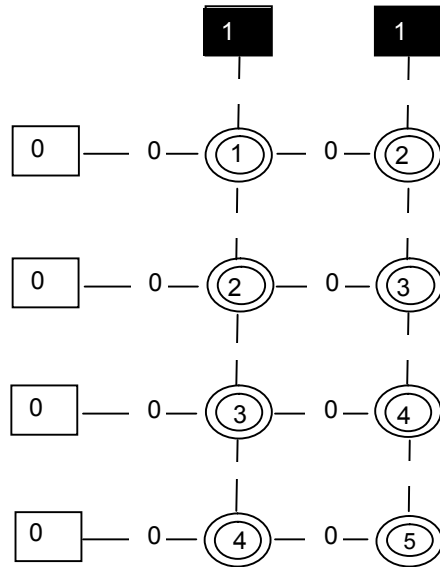




(b)



(c)



(d)

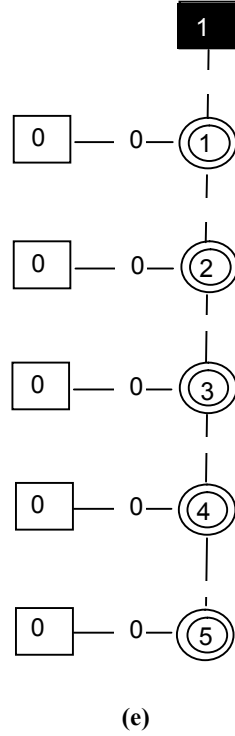


Fig. 3. The individual ROBDDs used in the computation of $R(k, 5, p)$, $1 \leq k \leq 5$. For simplicity, split (rather than combined) leaf nodes of 0 and 1 are used

5 Conclusions

We demonstrate algebraically and visually that the AR algorithm for the computing of k-out-of-n reliability is simply the particular ROBDD algorithm for such a computation. While ROBDD algorithms are generally intractable (of exponential complexity), this particular ROBDD algorithm is of quadratic temporal complexity. The results obtained herein are immediately extendible to systems generalizing the k-out-of-n system such as the (single-) threshold systems (also called weighted k-out-of-n systems), the double-threshold systems, the vector-threshold systems and k-to-l-out-of-n systems.

Competing Interests

Authors have declared that no competing interests exist.

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