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Solvability of nonlinear Klein-Gordon equation by Laplace Decomposition Method

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In this study, Adomian Decomposition Method (ADM), Modified Decomposition Method (MD) and Laplace Decomposition Method (LDM) were used in solving nonlinear Klein-Gordon equation. It can be easily concluded that these three methods yielded exactly the same results.

Key words: Adomian Decomposition Method, Modified Decomposition Method, Laplace Decomposition Method, Klein-Gordon equation and Noise terms phenomena.

INTRODUCTION

A wide variety of physically significant problems such as nonlinear Klein-Gordon equation, modeled by linear and nonlinear partial differential equations has been the focus of extensive studies for the last decades. A huge number of research and investigations have been invested in these scientific applications.

Several approaches such as the characteristics method, spectral methods and perturbation techniques have been extensively used to examine these problems (Wazwaz, 2009). Solving of nonlinear equations using Adomian decomposition method (ADM) has been done in Wazwaz 2009; 2006; El-Wakil et al. 2006; Adomian 1994;1984;1986; Abassy et al. 2007; 2004; Cherruault 1990; Lesnic 2006; 2007; Wazwaz 2001; Mohammed and Tarig 2013; 2014 and modified decomposition method (MD) in Mohammed and Tarig 2013; 2014.

The aim of this paper is in two folds: firstly, to solve the nonlinear Klein-Gordon equation via LDM, ADM and MD. Secondly, to show these three methods yielded exactly the same result.

As we know the nonlinear Klein-Gordon, equation comes from quantum field theory and describes nonlinear wave interaction. The nonlinear Klein-Gordon equation in its standard form is

$$u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) + F(u(x,t)) = h(x,t) \quad (1)$$

Subject to the initial conditions

$$u(x,0) = f(x)\ ,\ u_t(x,0) = g(x) \quad (2)$$

Where a is a constant, $h(x,t)$ is a source term and

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$F(u(x,t))$ is a nonlinear function of $u(x,t)$.

In this work, the noise terms phenomenon was used (Wazwaz, 2009), which, provides a major advantage in that it demonstrates a fast convergence of the solution. It is important to note that the noise terms phenomenon, may appear only for inhomogeneous partial differential equations; in addition, this phenomenon is applicable to all inhomogeneous PDEs of any order. The noise terms, if existed in the components u_0 and u_1 will provide, in general, the solution in a closed form with only two successive iterations.

Solution of Nonlinear Klein –Gordon Equation by ADM

The decomposition method will be employed. The nonlinear term $F(u(x,t))$ will be equated to the infinite series of Adomian polynomials (Adomian, 1994). In an operator form Equation (1) given by,

$$L_t(x,t) = L_x(x,t) - au(x,t) - F\big(u(x,t)\big) + h(x,t) \tag{3}$$

Where $L_t = \frac{\partial^2}{\partial t^2}$, $L_x = \frac{\partial^2}{\partial t^2}$ and $L_t^{-1} = \int_0^t \int_0^t (.)\,dtdt$

Applying L_t^{-1} to both sides of (3) and using the initial conditions to obtain,

$$u(x,t) = f(x) + tg(x) + L_t^{-1}\big(h(x,t)\big)$$

$$+L_t^{-1}\big(u_{xx}(x,t) - au(x,t)\big) - L_t^{-1}\Big(F\big(u(x,t)\big)\Big) \tag{4}$$

Using the decomposition series for the linear term by

$$u(x,t) = \sum_0^{\infty} u_n(x,t).$$

and the infinite series of Adomian polynomials for the nonlinear term $F(u(x,t))$ by

$$F\big(u(x,t)\big) = \sum_0^{\infty} A_n(x,t)$$

Where A_n are Adomian polynomials which calculated by

$$A_n = \frac{1}{n!}\frac{d^n}{d\alpha^n}\left[F\left(\sum_{n=0}^{\infty} \alpha^i u_i\right)\right]_{\alpha=0} \quad , \; n = 0,1,2,\ldots$$

We obtain the recursive relation:

$$u_0(x,t) = f(x) + tg(x) + L_t^{-1}\big(h(x,t)\big)$$

$$u_{k+1}(x,t) = L_t^{-1}\big(u_{k_{xx}}(x,t) - u_k(x,t)\big) - L_t^{-1}(A_k) \quad , \; k \geq 0 \tag{5}$$

that leads to:

$$u_0(x,t) = f(x) + tg(x) + L_t^{-1}\big(h(x,t)\big)$$

$$u_1(x,t) = L_t^{-1}\big(u_{0_{xx}}(x,t) - u_0(x,t)\big) - L_t^{-1}(A_0)$$

$$u_2(x,t) = L_t^{-1}\big(u_{1_{xx}}(x,t) - u_1(x,t)\big) - L_t^{-1}(A_1) \tag{6}$$

This completes the determination of the first few components of the solution. Based on this determination, the solution in a series form is readily obtained. In many cases, a closed form solution obtained conductively.

Example

Given the following nonlinear Klein-Gordon equation:

$$u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2 \;,\; u(x,0) = 1 \;,\; u_t(x,0) = x, \tag{7}$$

following the discussion presented above, we find:

$$u_0 = 1 + xt + \frac{xt^3}{3!} + \frac{x^2t^4}{12}$$

$$u_1 = \cdots - \frac{t^2}{2!} - \frac{xt^3}{3!} - \frac{x^2t^4}{12} - \cdots$$

Canceling the noise terms $\frac{xt^3}{3!}$ and $\frac{x^2t^4}{12}$ from the component u_0, and verifying that the remaining non-canceled terms (from the component u_0) satisfies the Equation (7), then the exact solution is

$$u(x,t) = 1 + xt \tag{8}$$

Solution of Nonlinear Klein –Gordon Equation by MD

In an operator form Equation (1) becomes

$$L_t(x,t) = L_x(x,t) - au(x,t) - F\big(u(x,t)\big) + h(x,t) \tag{9}$$

Where $L_t = \frac{\partial^2}{\partial t^2}$, $L_x = \frac{\partial^2}{\partial t^2}$ and $L_t^{-1} = \int_0^t \int_0^t (.)dtdt$

The modified decomposition method suggests that

$$u(x,t) = \sum_{m=0}^{\infty} a_m(x) t^m$$

$$h(x,t) = \sum_{r=0}^{\infty} r_m(x) t^m$$

$$F[u(x,t)] = \sum_{m=0}^{\infty} A_m(x) t^m$$

Operating with L_t^{-1} on both sides of Equation (9) subject to Equation (2) and using the above assumption, we obtain:

$$\sum_{m=0}^{\infty} a_m(x) t^m = f(x) + t\,g(x) + \sum_{m=0}^{\infty} r_m(x) \frac{t^{m+2}}{(m+1)(m+2)} + \frac{\partial^2}{\partial x^2} \sum_{m=0}^{\infty} a_m(x) \frac{t^{m+2}}{(m+1)(m+2)} - a\sum_{m=0}^{\infty} a_m(x) \frac{t^{m+2}}{(m+1)(m+2)} - \sum_{m=0}^{\infty} A_m(x) \frac{t^{m+2}}{(m+1)(m+2)} \quad (10)$$

Let $m = m-2$ be on the right side of Equation (10) and equate the coefficients of like power of t on both sides, we get:

$$\sum_{m=0}^{\infty} a_m(x) t^m = f(x) + g(x)t + \sum_{m=2}^{\infty} r_{m-2}(x) \frac{t^m}{m(m+1)} + \frac{\partial^2}{\partial x^2} \sum_{m=2}^{\infty} a_{m-2}(x) \frac{t^m}{m(m-1)} - a\sum_{m=2}^{\infty} a_{m-2}(x) \frac{t^m}{m(m-1)} - \sum_{m=2}^{\infty} A_{m-2}(x) \frac{t^m}{m(m-1)} \quad (11)$$

And then, the recurrence relation given by:

$$a_0(x) = f(x)\ ,\ \ a_1(x) = g(x)$$

$$a_m(x) = \frac{1}{m(m-1)}\left[r_m(x) + (a_{m-2}(x))_{xx} - a(a_{m-2}(x)) - A_{m-2}(x)\right] \quad ;\ m \geq 2$$

Having determined the coefficients, $a_m(x)$ the solution $u(x,t)$ in a series form, follow immediately.

Example:

Consider the initial nonlinear Klein-Gordon problem (7)

$$u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2,\quad u(x,0) = 1,\quad u_t(x,0) = x$$

Following the previous discussion, we find:

$$a_0(x) = 1\ ,\quad a_1(x) = x$$

$$a_m(x) = \frac{1}{m(m-1)}\left[r_{m-2}(x) + a_{m-2}(x) + (a_{m-2}(x))_{xx} - A_{m-2}(x)\right],\ m \geq 2$$

Where

$$A_m(x) = \sum_{n=0}^{m} a_{m-n}(x).a_n(x)$$

$$Calculate:\ a_1, a_2, \ldots, a_k$$

$$A_0 = 1$$

$$a_2(x) = 0$$
$$A_1(x) = 2x$$
$$a_3(x) = 0$$
$$a_k(x) = 0,\ k \geq 2$$

The solution in a series form given by

$$u(x,t) = \sum_{m=0}^{\infty} a_m(x) t^m = 1 + xt + 0 + 0 = 1 + xt$$

Solution of Nonlinear Klein –Gordon Equation by LDM

Here the LDM will be implemented to Klein-Gordon equation .To illustrate the method consider the general form of Klien-Gordon Equation (1) subject to the initial condition (2) and applying Laplace transform (denoted throughout this paper by L) on both sides of Equation (1) , yields:

$$L[u_{tt}(x,t)] - L[u_{xx}(x,t)] + aL[u(x,t)] + L[F(u(x,t))] = L[h(x,t)] \quad (12)$$

Which gives

$$s^2L[u(x,t)] - sf(x) - g(x) = L[u_{xx}(x,t)] - aL[u(x,t)] + L[h(x,t)] - L[F(u(x,t))] \quad (13)$$

So,

$$L[u(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L[u_{xx}(x,t)] - \frac{a}{s^2}L[u(x,t)] + \frac{1}{s^2}L[h(x,t)] - \frac{1}{s^2}L[F(u(x,t))]$$

Secondly using the decomposition series for the linear term $u(x,t)$ and the infinite series of Adomian polynomials for the nonlinear term $F(u(x,t))$ which gives,

$$L[\sum_0^\infty u_n(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L[h(x,t)] + \frac{1}{s^2}L\left[\left(\sum_0^\infty u_n(x,t)\right)_{xx}\right] - \frac{a}{s^2}L\left[\sum_0^\infty u_n(x,t)\right] - \frac{1}{s^2}L\left[\sum_0^\infty A_n\right] \quad (14)$$

Where A_n is the Adomian Polynomials given by Equation (5). Then Equation (14) becomes

$$\sum_0^\infty L[u_n(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L[h(x,t)] + \frac{1}{s^2}\sum_{n=0}^\infty L[u_{n_{xx}}(x,t)]$$

$$-\frac{a}{s^2}\sum_{n=0}^\infty L[u_n(x,t)] - \frac{1}{s^2}\sum_{n=0}^\infty L[A_n]$$

This gives the recurrence relation,

$$L[u_0(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L[h(x,t)] \quad (15)$$

$$L[u_{k+1}(x,t)] = \frac{1}{s^2}\sum_0^\infty L[u_{k_{xx}}(x,t)] - \frac{a}{s^2}\sum_{n=0}^\infty L[u_k(x,t)] - \frac{1}{s^2}\sum_{n=0}^\infty L[A_k]; k \geq 0 \quad (16)$$

Applying the inverse Laplace transform to the Equation (16), then the required recurrence relation is immediately obtained which complete the solution

$$u(x,t) = \sum_{n=0}^\infty u_n(x,t).$$

Application 1:

Consider the nonlinear Klein-Gordon Equation (7), using Equation (16) subject to the initial condition, we get

$$L[u(x,t)] = \frac{1}{s^2}\left(s + x + \frac{x}{s^2} + \frac{2x}{s^3}\right) + \frac{1}{s^2}(L[u_{xx}] + L[u] - L[u^2]) \quad (17)$$

Secondly, using the decomposition series for the linear term $u(x,t)$ and the infinite series of Adomian polynomials for the nonlinear term u^2 which gives,

$$L\left[\sum_{n=0}^\infty u(x,t)\right] = \frac{1}{s^2}\left(s + x + \frac{x}{s^2} + \frac{2x}{s^3}\right) + \frac{1}{s^2}\left(L\left[\left(\sum_{n=0}^\infty u_n\right)_{xx}\right] + L\left[\left(\sum_{n=0}^\infty u_n\right)\right] - L\left[\sum_{n=0}^\infty A_n\right]\right) \quad (18)$$

$$\sum_{n=0}^\infty L[u_n(x,t)] = \frac{1}{s^2}\left(s + x + \frac{x}{s^2} + \frac{2x}{s^3}\right) + \frac{1}{s^2}\left(\sum_{n=0}^\infty L[(u_n)_{xx}] + \sum_{n=0}^\infty L[u_n] - \sum_{n=0}^\infty L[A_n]\right) \quad (19)$$

That leads to the recurrence relation below

$$L[u_0] = \frac{1}{s} + \frac{x}{s^2} + \frac{x}{s^4} + \frac{2x^2}{s^5}$$

$$L[u_{k+1}] = \frac{1}{s^2}\left(L[u_{k_{xx}}] + L[u_k] - L[A_k]\right) \quad , k \geq 0 \quad (20)$$

Calculate: u_0

Applying inverse Laplace transform on the first equation of Equation (20) which gives

$$u_0 = 1 + xt + \frac{xt^3}{3!} + \frac{2x^2t^4}{4!}$$

Calculate: u_1

From Equation (18):

$$A_0 = u_0{}^2 = 1 + xt + \frac{xt^3}{3!} + \frac{2x^2t^4}{4!} + xt + x^2t^2 + \frac{x^2t^4}{3!} + \frac{x^3t^5}{12} + \frac{xt^3}{3!} + \frac{x^2t^4}{3!} + \frac{x^2t^6}{3!^2} + \frac{x^3t^7}{(3!)(12)} + \frac{x^2t^4}{12} + \frac{x^3t^5}{12} + \frac{x^3t^7}{(3!)(12)} + \frac{x^4t^8}{(12)(12)}$$

From Equation (20):

$$L[u_1] = \frac{1}{s^2}L[u_{0_{xx}} + u_0 - A_0] = \frac{1}{s^2}L\left[\frac{t^4}{6} - \frac{x^2t^4}{3} - xt - x^2t^2 - \frac{x^3t^5}{6} - \frac{xt^3}{3!} - \frac{x^3t^7}{36} - \frac{x^2t^6}{36} - \frac{x^4t^8}{144}\right]$$

$$= \frac{1}{s^2}\left[\frac{4!}{6s^5} - \frac{4!\,x^2}{3s^5} - \frac{x}{s^2} - \frac{2x^2}{s^3} - \frac{5!\,x^3}{6s^6} - \frac{x}{s^4} - \frac{7!\,x^3}{36s^8} - \frac{6!\,x^2}{36s^7} - \frac{8!\,x^4}{144s^9}\right]$$

Applying inverse Laplace to obtain

$$u_1 = \frac{4t^6}{6!} - \frac{8x^2t^6}{6!} - \frac{xt^3}{3!} - \frac{2x^2t^4}{4!} - \frac{20x^3t^7}{7!} - \frac{xt^5}{5!} - \frac{144x^3t^9}{9!} - \frac{20x^2t^8}{8!} - \frac{280x^4t^{10}}{(10)!}$$

Canceling the noise terms $\frac{xt^3}{3!}$ and $\frac{2x^2t^4}{4!}$ from the component u_0 and verifying that the remaining non-canceled terms satisfies the equation, the exact solution

$$u(x,t) = 1 + xt \ .$$

Application 2:

Consider the nonlinear Equation

$$u_{tt} - u_{xx} + u^2 = 6x^3t - 6xt^3 + x^6t^6 \qquad (21)$$

Subject to the initial conditions

$$u(x,0) = u_t(x,0) = 0$$

Following the analysis presented above and using the given initial conditions, we obtain the recursive relation in the form

$$L[u_0] = \frac{6x^3}{s^4} - \frac{36x}{s^6} + \frac{6!\,x^6}{s^9}$$

$$L[u_{k+1}] = \frac{1}{s^2}\left(L[u_{k_{xx}}] - L[A_k]\right) \qquad ,k \geq 0 \qquad (22)$$

Calculate: u_0

Applying inverse Laplace transform on the first equation of (22) that leads to

$$u_0 = x^3t^3 - \frac{3xt^5}{10} + \frac{x^6t^8}{56}$$

Calculate: u_1

$$A_0 = {u_0}^2 = \left(x^3t^3 - \frac{3xt^5}{10} + \frac{x^6t^8}{56}\right)^2$$

From the second Equation (22)

$$L[u_1] = \frac{1}{s^2}\left(L[u_{0_{xx}}] - L[A_0]\right)$$

Applying inverse Laplace transform on the last equation leads to

$$u_1 = \frac{3xt^5}{10} + \frac{x^4t^{10}}{168} - \frac{x^6t^8}{56} + \frac{x^4t^{10}}{300} - \frac{11!\,x^6t^{13}}{(56)13!} + \cdots$$

By canceling the noise terms $-\frac{3xt^5}{10}$ and $\frac{x^6t^8}{56}$ from the component u_0 and verifying that the remaining non-canceled terms of u_0 satisfies Equation (21), we find that the exact solution is given by

$$u(x,t) = x^3t^3$$

Conclusion

In this paper, we introduced Klein-Gordon equation, and solved it by using ADM, and MD then applied LDM .Clearly, these three methods are very effective, it accelerates the solutions. If we compare it with the other methods, it will be the best. In addition, the LDM may give the exact solutions for nonlinear PDEs. Moreover, the noise terms may appear if the exact solution is a part of the 'theroth component (u_0)' .

Conflict of Interest

The author has not declared any conflict of interest.

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