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Surface Wave Echo in a Plasma Slab

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Author's contribution

This work was carried out in collaboration between both authors. Author HJL designed the study, performed the analysis, wrote the protocol, and wrote the first draft of the manuscript, and managed literature searches. Author YKL managed the analysis of the study and literature searches. Both authors read and approved the final manuscript.

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ABSTRACT

Plasma echo theory is revisited, and we apply it to a plasma slab bounded by a vacuum. Spatial echoes in a slab plasma are investigated by calculating the electric field produced by external charges and satisfying the boundary conditions at the interfaces. We determine the echo spots associated with the symmetric mode of the surface wave in the slab. Naturally, in the course of development, the dispersion relation of the electrostatic surface plasma wave in a slab geometry is derived kinetically by satisfying the specular reflection boundary condition for the distribution function. We show that echoes can occur at various spots. The diversity of echo occurrence spots is due to the boundary terms, and appears to be owing to the reflections of the waves from the interface.

Keywords: Echo; plasma slab; boundary condition.

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1 INTRODUCTION

The existence of echoes in plasmas have long been known theoretically [1,2] as well as experimentally [3]. The theoretical investigations of the plasma echoes are confined mainly to echoes in an infinite plasma. The theory of the echoes for bounded plasmas are rather few. In this work, we investigate the plasma echoes in a slab which has two interfaces with vacuum. The interfaces act as the surfaces that reflect off the waves generated inside.

Let us briefly review the basic nonlinear mechanism that results in echoes. The echo phenomena is the result of a quadratic interaction of the two primary waves launched by two external charges at different locations (spatial echoes) or different times (temporal echoes). The linear response function (the dielectric function) of the plasma in the Fourier space has a singularity at $\omega = kv$ (in addition to other singularities). In its inversion to (x, t) variables, this singularity modulates the distribution function , $f(k, \omega, v)$, with the exponential phase $e^{ik(x-vt)}$ or $e^{-i\omega(t-x/v)}$. This term is called the free streaming term or the ballistic term since $x = vt$ is the characteristics of the Vlasov equation for a free particle. This rapidly modulating exponential phase makes the $f(x, v, t)$ more and more oscillatory as t or x increases, and consequently, $\int f dv$ will become vanishingly small due to almost complete *phase mixing*. Therefore, in the first order, the phase mixing obliterates any appreciable effect on the macroscopic variable such as density perturbation. However, the second order distribution function which is a product of two first order distribution functions is not phase-mixed when or where the condition for a constructive interference is met, thereby the second order electric field does not vanish, resulting in an echo. It is evident from the expression for the product of two free-streaming exponentials $e^{ik_1(x_1-vt_1)}e^{ik_2(x_2-vt_2)}$ that a constructive interference can result in at a certain time (temporal echo) or a certain spot (spatial echo) such that $k_1x_1 + k_2x_2 = v(k_1t_1 + k_2t_2)$.

Spatial echoes in a semi-bounded plasma were theoretically investigated in a static situation where the non-propagating electric field and the distribution function vary along the x direction which is perpendicular to the plasma-vacuum interface [4,5]: $E = E(x, t)$ and $f = f(x, v, t)$, where $x > 0$ ($x < 0$) is the plasma (vacuum) region. In this case, the corresponding Vlasov equation takes the form of a first order differential equation, and can be solved by satisfying the specular reflection boundary condition at the interface $x = 0 : f(v, 0) = f(-v, 0)$ [6]. This differential equation approach with the specular reflection boundary condition for a semi-bounded plasma has been shown to be entirely equivalent with the Fourier transform development provided that the $E(x)$ is extended into the region $x < 0$ in an odd function manner, $E(x) = -E(-x)$ [5,7]. Recently Lee and Lee [8] investigated the spatial echoes associated with a propagating surface wave in a semi-bounded plasma. In that work [8], the authors obtained the echo spots in a more general circumstance; namely, in terms of both x and z- coordinates.

The salient feature in solving the Vlasov equation in a bounded plasma is the odd function extension of $E(x)$ that gives rise to a surface term at $x = 0$ in the Fourier transform of the Poisson equation. The surface term plays an important role in the determination of the echo spots. The odd function discontinuity of $E(x)$ at the interface is characteristic of a bounded plasma. The diversity of echo spots [5,8] is due to the surface term. Physically, the surface term manifests the reflection of the electric field at the boundary.

In this work, we investigate spatial echoes in a slab plasma. Generally, analysis of surface waves in a slab geometry contains extra complication as compared to the case of semi-bounded plasmas because we have two interfaces, $x = 0$ and $x = L$, on both of which the specular reflection boundary condition should be satisfied. The extension from the case of a semi-bounded geometry is not trivial, and the determination of echo occurrence spots in the slab is not straightforward. Earlier Lee and Lim [9] derived the linear dispersion relation of the surface wave in a slab plasma. In the present problem of plasma echoes, the governing equation has an additional term, a term of the external charges which launches echoes. Therefore the analytical development of the determination of the electric field in a slab is parallel to the linear analysis in Ref. [9].

In section II, we show how the odd function extension of $E_x(x)$ emerges from the invariant argument of the Vlasov equation under the simultaneous reflections $x \to -x$ and $v \to -v$, thereby satisfying the specular reflection condition at $x = 0$ and $x = L$. In section III, the plasma electric field is determined in terms of the external charges by satisfying the electric continuity equations across the interfaces $x = 0$ and $x = L$. An important feature of the electric field in a slab is the presence of the cosine series S (see Eq. (3.42)) which stems from the odd function discontinuity of $E_x(x)$ at $x = 0$ and $x = L$. [In a semi-bounded plasma, we have a single term, not a series.] It was shown by fluid equations that the surface wave in a slab has two modes; symmetric and anti-symmetric mode [10]. The kinetic surface wave also exhibits the two modes. In this work, we investigate only the echoes associated with the symmetric mode. Section IV determines the echo spots by calculating the second order electric field, the product of two first order electric fields. This section contains mainly mathematical details in carrying out the Fourier inversion integrals by contour integration , which leads to Eqs. (4.49) and (4.50), the echo spots. We show a mathematical finesse on how we can utilize the linear dispersion relation in Fourier inverting the aforementioned cosine series.

We have a diversity in the echo spots. The diversity of echo occurrence spots has been experimentally reported [11] and can be explained by the boundary terms, and appears to be due to the reflections of the waves off the interface. The identification of the echo spot associated with surface wave appears to be useful in experimental point of view [11].

2 FORMULATION OF THE PROBLEM

We consider a plasma consisting of electrons and stationary ions, the latter forming the uniform background. The plasma is assumed to occupy the slab $0 < x < L$. The regions $x < 0$ and $x > L$ are assumed to be vacuum. The perturbed electron distribution function $f(\mathbf{r}, \mathbf{v}, t)$ and the electric field $E(\mathbf{r}, t)$ will depend on x and z-coordinates with the y coordinate ignored since y direction has a translational invariance. The basic state is a plasma surface wave propagating in the slab with the phasor $exp(ik_x x + ik_z z - i\omega t)$ with external charges as prescribed in Eq. (2.3) below. We have the nonlinear Vlasov equation and the Poisson equation to describe the electrostatic perturbation:

$$\frac{\partial}{\partial t} f(\mathbf{v}, \mathbf{r}, t) + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{e}{m} \mathbf{E}(\mathbf{r}, t) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad (2.1)$$

$$with\ \mathbf{r} = \hat{\mathbf{x}} x + \hat{\mathbf{z}} z, \ \ \mathbf{v} = \hat{\mathbf{x}} v_x + \hat{\mathbf{z}} v_z, \quad \mathbf{E} = \hat{\mathbf{x}} E_x + \hat{\mathbf{z}} E_z$$

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 4\pi \left(-e \int d^2 v f + \rho_0(x, z, t) \right) \quad (2.2)$$

where f is two-dimensional distribution function, and ρ_0 represents the external charges:

$$\rho_0(x, z, t) = \rho_1 e^{i\omega_1 t} \delta[k_0(x - L_1)] \delta[k_0(z - \zeta_1)] + \rho_2 e^{i\omega_2 t} \delta[k_0(x - L_2)] \delta[k_0(z - \zeta_2)] \quad (2.3)$$

where k_0 is introduced to make the argument of the δ–function dimensionless. We solve the simultaneous equations (2.1) and (2.2) for a given $\rho_0(x, z, t)$ as prescribed by Eq. (2.3). In mathematical terms, we have an inhomogeneous system, driven by the source term in Eq. (2.3). The responses f and $\mathbf{E}$ should be determined by ρ_0.

The kinetic equation is supplemented by the kinematic boundary condition which we assume to be the specular reflection condition

$$f(v_x, x = 0) = f(-v_x, x = 0), \ \ f(v_x, x = L) = f(-v_x, x = L) \quad (2.4)$$

Assuming that the external perturbation is small, we solve Eqs. (2.1) and (2.2) by successive approximation. First, the linear solution of Eq. (2.1) will be obtained for f with the boundary condition (2.4). Substituting this solution in Eq. (2.2) yields an integral equation for the electric field which is solved by Fourier transform. Then the linear solution will be used to obtain the higher order solutions. We work only up to the second order. The higher order distribution function should also satisfy the boundary condition (2.4). The electric field should satisfy the electric boundary conditions: the normal component of the electric displacement $D_x(x)$ and the tangential electric field $E_z(x)$ are continuous across the interface.

The specular reflection boundary conditions in Eq. (2.4) can be satisfied by extending the electric field components by the recipe [9]

$$E_x(-x) = -E_x(x), \quad E_x(2L - x) = -E_x(x) \tag{2.5}$$

$$E_z(-x) = E_z(x), \quad E_z(2L - x) = E_z(x) \tag{2.6}$$

The functions $E_x(x)$ and $E_z(x)$ as defined by Eqs. (2.5) and (2.6) are plotted in Figures 3 and 4 in Ref. [12] with a linear profile assumed (interchange letters x nd z therein). $E_x(x)$ and $E_z(x)$ are piecewise continuous with discontinuities at $x = \pm nL$ (n: integer). The jump of $E_x(x)$ at the discontinuities $x = \pm nL$ should be carefully accounted for in Fourier transforming $\partial E_x/\partial x$. Let us Fourier transform the above equations with respect to t and z. In this work, the Fourier transforms are defined by

$$f(k, \omega) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt f(x, t) e^{-ikx + i\omega t}$$

$$f(x, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(k, \omega) e^{ikx - i\omega t}$$

Equation (2.1) is transformed to the form

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}) f(\mathbf{v}, \mathbf{k}, \omega) - \frac{e}{m} \int \frac{d\omega'}{2\pi} \int$$

$$\frac{d^2 k'}{(2\pi)^2} \mathbf{E}(\mathbf{k} - \mathbf{k}', \omega - \omega') \cdot \frac{\partial}{\partial \mathbf{v}} f(\mathbf{v}, \mathbf{k}', \omega') = 0 \tag{2.7}$$

where $\mathbf{k} = \hat{\mathbf{x}} k_x + \hat{\mathbf{z}} k_z$.

In Fourier transforming Eq. (2.2), care should be exercised in regard to the term $\partial E_x/\partial x$:

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dz e^{-ik_x x - ik_z z} \frac{\partial}{\partial x} E_x(x, z, \omega) = \int_{-\infty}^{\infty} dx e^{-ik_x x} \frac{\partial}{\partial x} E_x(x, k_z, \omega) =$$

$$ik_x E_x(k_x, k_z, \omega) + \Big[E_x(x, k_z, \omega) e^{-ik_x x} \Big] \Big(\Big|_{0^+}^{L^-} + \Big|_{L^+}^{2L^-} + \Big|_{2L^+}^{3L^-} + \cdots \Big)$$

$$- \Big[E_x(x, k_z, \omega) e^{-ik_x x} \Big] \Big(\Big|_{0^-}^{-L^+} + \Big|_{-L^-}^{-2L^+} + \Big|_{-2L^-}^{-3L^+} + \cdots \Big) \tag{2.8}$$

where $\pm nL^{\pm} = \pm nL \pm \epsilon$ $(n = 1, 2, \cdots)$ with ϵ positive infinitesimal. Then, Fourier transform of Eq. (2.2) becomes

$$i\mathbf{k} \cdot \mathbf{E}(\mathbf{k}, \omega) + N(k_z, k_x, \omega) = 4\pi \left[-e \int f(\mathbf{k}, \mathbf{v}, \omega) d^2 v + \rho_0(\mathbf{k}, \omega) \right] \tag{2.9}$$

$$N(k_z, k_x, \omega) = A(cosk_x \epsilon + cosk_x 2L^- + cosk_x 2L^+ + cosk_x 4L^- + cosk_x 4L^+ + \cdots)$$

$$+ B(cosk_x L^- + cosk_x L^+ + cosk_x 3L^- + cosk_x 3L^+ + \cdots)$$

$$= 2A\Big(\frac{1}{2} + cos2k_x L + cos4k_x L + \cdots \Big) + 2B(cosk_x L + cos3k_x L + \cdots)$$

where $A = 2E_x(0^-, k_z, \omega)$, $\quad B = 2E_x(L^-, k_z, \omega)$. In writing the last line, we put $L^+ = L^- = L$, and $\epsilon = 0$. The $N(k_x, k_z, \omega)$ term in Eq. (2.9) is characteristic of a slab plasma and responsible for the diversity of surface wave echoes, as compared with an infinite plasma.

Equation (2.3) gives

$$\rho_0(\mathbf{k},\omega) = \frac{2\pi}{k_0^2}\rho_1\delta(\omega+\omega_1)e^{-ik_xL_1}\; e^{-ik_z\zeta_1} + 1 \rightarrow 2 \tag{2.10}$$

Equations (2.7) and (2.9) constitute a set of nonlinear simultaneous equations . We solve the set of equations by successive approximations in terms of perturbation series:

$$f(\mathbf{k},\mathbf{v},\omega) = f_0(\mathbf{v}) + f^{(1)}(\mathbf{k},\mathbf{v},\omega) + f^{(2)}(\mathbf{k},\mathbf{v},\omega) + \cdots$$

$$\mathbf{E}(\mathbf{k},\omega) = \mathbf{E}^{(1)}(\mathbf{k},\omega) + \mathbf{E}^{(2)}(\mathbf{k},\omega) + \cdots$$

Breaking down Eqs. (2.7) and (2.9) order by order, we have

$$-i(\omega - \mathbf{k}\cdot\mathbf{v})f^{(1)}(\mathbf{k},\mathbf{v},\omega) = \frac{e}{m}\,\mathbf{E}^{(1)}(\mathbf{k},\omega)\cdot\frac{df_0}{d\mathbf{v}} \tag{2.11}$$

$$i\mathbf{k}\cdot\mathbf{E}^{(1)}(\mathbf{k},\omega) + N(k_z,k_x,\omega) = 4\pi\left[-e\int f^{(1)}(\mathbf{k},\mathbf{v},\omega)d^2v + \rho_0(\mathbf{k},\omega)\right] \tag{2.12}$$

$$-i(\omega - \mathbf{k}\cdot\mathbf{v})f^{(2)}(\mathbf{k},\mathbf{v},\omega) = \frac{e}{m}\,\mathbf{E}^{(2)}(\mathbf{k},\omega)\cdot\frac{df_0}{d\mathbf{v}} + \frac{e}{m}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\times$$

$$\int_{-\infty}^{\infty}\frac{dk_x'}{2\pi}\int_{-\infty}^{\infty}\frac{dk_z'}{2\pi}\mathbf{E}^{(1)}(\mathbf{k}-\mathbf{k}',\omega-\omega')\cdot\frac{\partial}{\partial\mathbf{v}}f^{(1)}(\mathbf{k}',\omega',\mathbf{v}) \tag{2.13}$$

$$i\,\mathbf{k}\cdot\mathbf{E}^{(2)}(\mathbf{k},\omega) = -4\pi e\int f^{(2)}(\mathbf{k},\mathbf{v},\omega)d^2v \tag{2.14}$$

3 LINEAR SOLUTION

Equations (2.11) and (2.12), and $\nabla\times\mathbf{E} = 0$ give

$$\mathbf{E}^{(1)}(\mathbf{k},\omega) = \frac{i\mathbf{k}}{k^2\,\varepsilon(\mathbf{k},\omega)}[N(k_z,\omega) - 4\pi\rho_0(\mathbf{k},\omega)] \tag{3.1}$$

where

$$\varepsilon(\mathbf{k},\omega) = 1 + \frac{\omega_p^2}{k^2}\int d^2v\,\frac{\mathbf{k}\cdot\frac{df_0}{d\mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}} \tag{3.2}$$

is the dielectric function (ω_p is the plasma frequency). The constants A and B in N are determined from the electric boundary conditions as shown in the following. Then, $\mathbf{E}^{(1)}$ is determined entirely in terms of the external charges. One of the electric boundary conditions is the continuity of the normal component of electric displacement, $D_x(x)$. The latter in the plasma side can be obtained from Eq. (2.12). By definition, $D_x(\mathbf{k},\omega) = E_x(\mathbf{k},\omega) + \frac{4\pi i}{\omega}J_x(\mathbf{k},\omega)$ where J is the current, $J_x(\mathbf{k},\omega) = -e\int d^2v\; v_x\; f(\mathbf{k},\omega,\mathbf{v})$. We calculate

$$\frac{4\pi i}{\omega}J_x = \frac{4\pi i}{\omega}(-e)\int v_x f d^2v = \frac{\omega_p^2}{\omega}\int d^2v v_x\,\frac{E_j\frac{\partial f_0}{\partial v_j}}{\omega - \mathbf{k}\cdot\mathbf{v}}$$

where we used Eq. (2.11). The above quantity equals to $(\varepsilon - 1)E_x$. Thus we have $D_x = \varepsilon E_x$. This equality can be most easily proved by assuming f_0 a Maxwellian. Use $\frac{\partial f_0}{\partial v_j} = -\frac{T}{m} v_j f_0$ and $\mathbf{E}\cdot\mathbf{v} = \frac{E_x}{k_x}\mathbf{k}\cdot\mathbf{v}$ to simplify the last term, and thus

$$\frac{4\pi i}{\omega} J_x = -E_x \frac{\omega_p^2}{\omega}\frac{T}{m}\int \frac{v_x}{k_x}\ \frac{\mathbf{k}\cdot\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}} f_0\ d^2v$$

Put $\frac{\mathbf{k}\cdot\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}} = -1 + \frac{\omega}{\omega-\mathbf{k}\cdot\mathbf{v}}$. (-1)–term vanishes upon integration, and we have

$$\frac{4\pi i}{\omega} J_x = -E_x \frac{\omega_p^2}{k_x}\frac{T}{m}\int \frac{v_x f_0}{\omega - \mathbf{k}\cdot\mathbf{v}} d^2v = E_x \frac{\omega_p^2}{k_x}\int \frac{\frac{\partial f_0}{\partial v_x}}{\omega - \mathbf{k}\cdot\mathbf{v}} d^2v$$

$$= -E_x\ \omega_p^2 \int \frac{f_0\ d^2v}{(\omega - \mathbf{k}\cdot\mathbf{v})^2} = E_x \frac{\omega_p^2}{k^2}\int d^2v\ \frac{\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}} \quad q.e.d.$$

Using the above result, we obtain

$$D_x^{(1)}(\mathbf{k},\omega) = \varepsilon(\mathbf{k},\omega)\ E_x^{(1)}(\mathbf{k},\omega) = \frac{ik_x}{k^2}[N - 4\pi\rho_0(\mathbf{k},\omega)] \tag{3.3}$$

Also we should invert

$$E_z^{(1)}(\mathbf{k},\omega) = \frac{ik_z}{k^2\ \varepsilon(\mathbf{k},\omega)}[N(k_z,\omega) - 4\pi\rho_0(\mathbf{k},\omega)] \tag{3.4}$$

To invert Eq. (3.3), we write

$$D_x^{(1)}(x,k_z,\omega) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\ e^{ik_x x}\frac{2ik_x}{k^2}\ \Big[A(\ \frac{1}{2} + cos2k_xL + cos4k_xL + \cdots)$$

$$+B(cosk_xL + cos3k_xL + \cdots) - 2\pi\rho_0(\mathbf{k},\omega)\Big] \tag{3.5}$$

It is important to integrate term by term, and sum over the integrated terms. Let us consider a term

$$\int_{-\infty}^{\infty} \frac{k_x dk_x}{k^2} e^{ik_x x}\ cos\ nk_xL \quad (n = 1, 2, 3, \cdots)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{k_x dk_x}{k^2} e^{ik_x x}\ (e^{ink_xL} + e^{-ink_xL}) = i\int_{-\infty}^{\infty} \frac{k_x dk_x}{k^2}\ e^{ink_xL} sin\ k_x x \tag{3.6}$$

To obtain the last equality, we changed variable $k_x \to -k_x$. Therefore, Eq. (3.5) takes the form

$$D_x^{(1)}(x,k_z,\omega) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{k_x dk_x}{k^2}\ \Big[\frac{A}{2}e^{ik_x x} + Aisin\ k_x x\ (e^{2ik_xL} + e^{4ik_xL} + \cdots)$$

$$+iBsin\ k_x x(e^{ik_xL} + e^{3ik_xL} + \cdots) - 2\pi\rho_0(\mathbf{k},\omega)e^{ik_x x}\Big] \tag{3.7}$$

Likewise, we can write the inversion integral of Eq. (3.4) in the form

$$E_z^{(1)}(x,k_z,\omega) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{k_z dk_x}{k^2\varepsilon}\ \Big[\frac{A}{2}e^{ik_x x} + Acos\ k_x x\ (e^{2ik_xL} + e^{4ik_xL} + \cdots)$$

$$+Bcos\ k_x x\ (e^{ik_xL} + e^{3ik_xL} + \cdots) - 2\pi\rho_0(\mathbf{k},\omega)e^{ik_x x}\Big] \quad (3.8)$$

Equations (3.7) and (3.8) are the plasma solutions in which A and B, the constants, should be determined from the boundary conditions. In particular, the boundary values are as follows.

$$D_x^{(1)}(0,k_z,\omega) = -\frac{A}{2} - 2i\int_{-\infty}^{\infty}\frac{dk_xk_x}{k^2}\rho_0(\mathbf{k},\omega) \quad (3.9)$$

At $x = L$, Eq. (3.7) gives quite simple result. After successive cancellations, the coefficient of A becomes zero. The coefficient of B inside the large bracket reduces to $-\frac{1}{2}$. Therefore we get

$$D_x^{(1)}(L,k_z,\omega) = \frac{B}{2} - 2i\int_{-\infty}^{\infty}\frac{dk_xk_x}{k^2}\rho_0(\mathbf{k},\omega)e^{iLk_x} \quad (3.10)$$

Equation (3.8) immediately gives

$$E_z^{(1)}(0,k_z,\omega) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{dk_xk_z}{k^2\varepsilon}\Big[AS_1 + BS_2 - 2\pi\rho_0(\mathbf{k},\omega)\Big] \quad (3.11)$$

$$where\ \ S_1 = \frac{1}{2} + e^{2iLk_x} + e^{4iLk_x} + \cdots,\quad S_2 = e^{iLk_x} + e^{3iLk_x} + \cdots \quad (3.12)$$

At $x = L$, the following interesting relation (in view of Eq. (3.11)) can be easily proved:

$$E_z^{(1)}(L,k_z,\omega) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{dk_xk_z}{k^2\varepsilon}\Big[AS_2 + BS_1 - 2\pi\rho_0(\mathbf{k},\omega)e^{iLk_x}\Big] \quad (3.13)$$

Next, we should obtain the vacuum solutions outside the slab. We have $\nabla^2\phi(x,z) = 0$, with $\mathbf{E} = -\nabla\phi$. Fourier transforming this equation with respect to z, the vacuum equation reads

$$\frac{\partial^2}{\partial x^2}\phi(x,k_z) - k_z^2\phi = 0 \quad (3.14)$$

which is solved by the functions $\phi \sim exp(\pm k_z x)$. We obtain

i) in the region $x > L$:

$$E_x(x,k_z) = A'k_ze^{-k_zx},\ \ E_z(x,k_z) = -iA'k_ze^{-k_zx} \quad (3.15)$$

ii) in the region $x < 0$:

$$E_x(x,k_z) = -B'k_ze^{k_zx},\ \ E_z(x,k_z) = -iB'k_ze^{k_zx} \quad (3.16)$$

where A' and B' are constants to be determined from the boundary conditions at the interfaces. Using the preceding equations, we can write the continuity equations. Continuity of D_x at $x = 0$ yields

$$-\frac{A}{2} - \frac{4\pi i}{k_0^2}\rho_1\delta(\omega+\omega_1)e^{-ik_z\zeta_1}\int_{-\infty}^{\infty}\frac{k_xdk_x}{k^2}e^{-ik_xL_1} + 1 \to 2 = -B'k_z \quad (3.17)$$

Continuity of D_x at $x = L$ yields

$$\frac{B}{2} - \frac{4\pi i}{k_0^2}\rho_1\delta(\omega+\omega_1)e^{-ik_z\zeta_1}\int_{-\infty}^{\infty}\frac{k_xdk_x}{k^2}e^{ik_x(L-L_1)} + 1 \to 2 = A'k_ze^{-k_zL} \quad (3.18)$$

Continuity of E_z at $x = 0$ yields

$$\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{k_zdk_x}{k^2\varepsilon}\Big[AS_1 + BS_2\Big] - \frac{4\pi i}{k_0^2}\rho_1\delta(\omega+\omega_1)e^{-ik_z\zeta_1}\int_{-\infty}^{\infty}\frac{k_zdk_x}{k^2\varepsilon}e^{-ik_xL_1} + 1 \to 2$$

$$= -ik_z B' \tag{3.19}$$

Continuity of E_z at $x = L$ yields

$$\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{k_z dk_x}{k^2\varepsilon}\Big[BS_1 + AS_2\Big] - \frac{4\pi i}{k_0^2}\rho_1\delta(\omega+\omega_1)e^{-ik_z\zeta_1}\int_{-\infty}^{\infty}\frac{k_z dk_x}{k^2\varepsilon}e^{ik_x(L-L_1)} + 1 \to 2$$

$$= -ik_z A'\, e^{-k_z L} \tag{3.20}$$

Equations (3.15)–(3.20) determine the four constants A, B, A', and B'. A' and B' are easily eliminated, and we have simultaneous equations for A and B:

$$A(\frac{1}{2} + I_1) + BI_2 = \xi_0 \tag{3.21}$$

$$AI_2 + B(\frac{1}{2} + I_1) = \xi_L \tag{3.22}$$

where

$$I_i = \frac{k_z}{\pi}\int_{-\infty}^{\infty}\frac{dk_x}{k^2\varepsilon}S_i \quad (i = 1, 2) \tag{3.23}$$

$$\xi_0 = \frac{4\pi}{k_0^2}\rho_1\delta(\omega+\omega_1)e^{-ik_z\zeta_1}\int_{-\infty}^{\infty}\frac{dk_x}{k^2}(\frac{k_z}{\varepsilon} - ik_x)e^{-ik_x L_1} + 1 \to 2 \tag{3.24}$$

$$\xi_L = \frac{4\pi}{k_0^2}\rho_1\delta(\omega+\omega_1)e^{-ik_z\zeta_1}\int_{-\infty}^{\infty}\frac{dk_x}{k^2}(\frac{k_z}{\varepsilon} + ik_x)e^{ik_x(L-L_1)} + 1 \to 2 \tag{3.25}$$

Equations (3.21) and (3.22) yield

$$A = \frac{1}{\Delta}\Big[(\frac{1}{2} + I_1)\xi_0 - I_2\xi_L\Big] \tag{3.26}$$

$$B = \frac{1}{\Delta}\Big[(\frac{1}{2} + I_1)\xi_L - I_2\xi_0\Big] \tag{3.27}$$

$$\Delta = (\frac{1}{2} + I_1)^2 - I_2^2 \tag{3.28}$$

Let us digress here to investigate dispersion relation of the surface wave that is implied in the above equations. When $\rho_1 = \rho_2 = 0$ ($\xi_0 = \xi_L = 0$), Eqs. (3.21) and (3.22) give the solvability condition,

$$\Delta = (\frac{1}{2} + I_1 + I_2)(\frac{1}{2} + I_1 - I_2) = 0 \tag{3.29}$$

Therefore, $\frac{1}{2} + I_1 = \pm I_2$ are the two dispersion relations, respectively for the symmetric and anti-symmetric mode in a plasma slab. We write Eq. (3.29) in the form

$$\frac{1}{2} + \frac{k_z}{\pi}\int_{-\infty}^{\infty}\frac{dk_x}{k^2\varepsilon}\Big[\frac{1}{2} \pm e^{ik_xL} + e^{2ik_xL} \pm e^{3ik_xL} + e^{4ik_xL} \pm \cdots\Big] = 0 \tag{3.30}$$

Although the series consisting of the exponential terms are not convergent as they are, they are convergent upon picking up the poles located in the upper k_x-plane from the denominator. Summing up the two series formally, the two dispersion relations are obtained in the form [9]

$$1 + \frac{k_z}{\pi}\int\frac{dk_x}{k^2\varepsilon}\Big[\frac{1 \pm e^{iLk_x}}{1 \mp e^{iLk_x}}\Big] = 0 \tag{3.31}$$

In the fluid limit, Eq. (3.31) yields the dispersion relation that agrees with the slab dispersion relations obtained from the fluid equations [9,10]. We will use the dispersion relation, $\frac{1}{2}+I_1=\pm I_2$, to simplify Eqs. (3.26) and (3.27). Let us write

$$A=\frac{1}{2I_2}[\xi_0(\frac{1}{2}+I_1)-\xi_L I_2]\left[\frac{1}{\frac{1}{2}+I_1-I_2}-\frac{1}{\frac{1}{2}+I_1+I_2}\right] \tag{3.32}$$

For the symmetric mode, we can put $\frac{1}{\frac{1}{2}+I_1+I_2}=\delta(\frac{1}{2}+I_1+I_2)$ and the other term is negligible. Therefore we obtain

$$A=\frac{1}{2}\,\frac{\xi_0+\xi_L}{\frac{1}{2}+I_1+I_2}=B \tag{3.33}$$

For the anti-symmetric mode, $\frac{1}{\frac{1}{2}+I_1-I_2}=\delta(\frac{1}{2}+I_1-I_2)$ and the other term is negligible. Therefore we obtain

$$A=\frac{1}{2}\,\frac{\xi_0-\xi_L}{\frac{1}{2}+I_1-I_2}=-B \tag{3.34}$$

In this work, we investigate the symmetric mode echoes. Let us calculate ξ_0 and ξ_L in Eqs. (3.24) and (3.25) by contour integration. The k_x-contour in ξ_0 (ξ_L) should wind the lower (upper) k_x-plane. The following results can be easily verified:

$$\int_{-\infty}^{\infty} dk_x\,\frac{k_z}{k^2}\,e^{-ik_xL_1}=\pi\left[H(Re\;k_z)e^{-L_1k_z}-H(-Re\;k_z)e^{L_1k_z}\right] \tag{3.35}$$

$$\int_{-\infty}^{\infty} dk_x\,\frac{k_x}{k^2}\,e^{-ik_xL_1}=-i\,\pi\left[H(Re\;k_z)e^{-L_1k_z}+H(-Re\;k_z)e^{L_1k_z}\right] \tag{3.36}$$

$$\int_{-\infty}^{\infty} dk_x\,\frac{k_z}{k^2}\,e^{ik_x(L-L_1)}=-\pi\left[H(-Re\;k_z)e^{k_z(L-L_1)}-H(Re\;k_z)e^{-k_z(L-L_1)}\right] \tag{3.37}$$

$$\int_{-\infty}^{\infty} dk_x\,\frac{k_x}{k^2}\,e^{ik_x(L-L_1)}=i\pi\left[H(-Re\;k_z)e^{k_z(L-L_1)}+H(Re\;k_z)e^{-k_z(L-L_1)}\right] \tag{3.38}$$

where $H(x)$ is the step function: $H(x)=1$ for $x>0$ or $H(x)=0$ for $x<0$. In evaluating $\xi_{0,L}$, the contribution from the pole at $\varepsilon=0$ can be neglected. Thus we obtain

$$\xi_0=\frac{4\pi^2}{k_0^2}\rho_1\delta(\omega+\omega_1)e^{-ik_z\zeta_1}\left[(\frac{1}{\varepsilon}-1)H(Re\;k_z)e^{-L_1k_z}-(\frac{1}{\varepsilon}+1)H(-Re\;k_z)e^{L_1k_z}\right]+ \\ +(1\to 2) \tag{3.39}$$

$$\xi_L=\frac{4\pi^2}{k_0^2}\rho_1\delta(\omega+\omega_1)e^{-ik_z\zeta_1}\left[-(\frac{1}{\varepsilon}+1)H(-Re\;k_z)e^{k_z(L-L_1)}+(\frac{1}{\varepsilon}-1)H(Re\;k_z)e^{-k_z(L-L_1)}\right] \\ +(1\to 2) \tag{3.40}$$

where $(1\to 2)$ means the preceding term with the subscript 1 replaced by subscript 2. For symmetric mode, the expression for $N(k_x,k_z,\omega)$ becomes (see the equation below Eq. (2.9))

$$N(k_z,k_x,\omega)=(\xi_0+\xi_L)(\frac{1}{2}+I_1+I_2)^{-1}S(k_xL)+(1\to 2) \tag{3.41}$$

$$with \quad S(k_x L) = \frac{1}{2} + cosk_x L + cos2k_x L + cos3k_x L + cos4k_x L + \cdots \tag{3.42}$$

In Eq. (3.41), $(\frac{1}{2}+I_1+I_2)^{-1}S(k_xL) = \delta(\frac{1}{2}+I_1+I_2)S(k_xL) = \big[S(k_xL)\big]_{\frac{1}{2}+I_1+I_2=0}$. In words, $S(k_xL)$ in Eq. (3.41) is calculated by means of the dispersion relation $\frac{1}{2}+I_1+I_2=0$. Keeping this in mind, the δ-function designation will be omitted. Thus the electric field in the slab, Eq. (3.1), is written as

$$\mathbf{E}^{(1)}(\mathbf{k},\omega) = \frac{i\mathbf{k}}{k^2\ \varepsilon(\mathbf{k},\omega)} P(1,\omega,k_z)\Big(R(1,k_z)S(k_x) - 2e^{-ik_xL_1}\Big) + 1 \to 2 \tag{3.43}$$

where

$$P(1,\omega,k_z) = \frac{4\pi^2}{k_0^2}\rho_1\delta(\omega+\omega_1)e^{-ik_z\zeta_1} \tag{3.44}$$

$$R(1,k_z) = (\frac{1}{\varepsilon}-1)H(Re\ k_z)(e^{-k_z(L-L_1)} + e^{-L_1k_z}) - (\frac{1}{\varepsilon}+1)H(-Re\ k_z)(e^{k_z(L-L_1)} + e^{L_1k_z}) \tag{3.45}$$

Equation (3.43) is the electric field in the slab, generated by the external charges and the plasma, satisfying the boundary conditions. This equation will be used to calculate the quadratic second order electric field in the next section.

4 THE SECOND ORDER SOLUTION AND ECHO OCCURRENCE

Next, we deal with the second order equations, Eqs. (2.10) and (2.11). Using Eq. (2.10) in Eq. (2.11) yields, owing to the electrostatic nature of $\mathbf{E}^{(2)}$,

$$\mathbf{E}^{(2)}(\omega,\mathbf{k}) = \frac{4\pi e^2}{m}\ \frac{\mathbf{k}}{k^2\varepsilon(\mathbf{k},\omega)}\int d^2v\ \frac{\mathbf{k}\cdot\mathbf{Q}}{(\omega-\mathbf{k}\cdot\mathbf{v})^2} \tag{4.1}$$

where $\mathbf{Q}$ stands for

$$\mathbf{Q}(\omega,\mathbf{k},\mathbf{v}) = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{dk'_x}{2\pi}\int_{-\infty}^{\infty}\frac{dk'_z}{2\pi}\mathbf{E}^{(1)}(\omega-\omega',\mathbf{k}-\mathbf{k}')f^{(1)}(\omega',\mathbf{k}',\mathbf{v}) \tag{4.2}$$

Substituting the first order solutions, Eqs. (2.8) and (3.5), into the above equations, we can write $\mathbf{E}^{(2)}$ in the form,

$$\mathbf{E}^{(2)}(\omega,\mathbf{k}) = \frac{-ie^3}{2\pi^2m^2}\ \frac{\mathbf{k}}{k^2\varepsilon(\mathbf{k},\omega)}\int\ \frac{d^2v}{(\omega-\mathbf{k}\cdot\mathbf{v})^2}\int dk'_z\int dk'_x\int d\omega'\times$$

$$\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{\varepsilon(\omega-\omega',\mathbf{k}-\mathbf{k}')(\mathbf{k}-\mathbf{k}')^2}\ \frac{\mathbf{k}'\cdot\frac{df_0}{d\mathbf{v}}}{k'^2\varepsilon(\omega',\mathbf{k}')(\omega'-\mathbf{k}'\cdot\mathbf{v})}$$

$$\times[\Gamma(1,\omega',\mathbf{k}')+1\to 2]\ [\Gamma(1,\omega-\omega',\mathbf{k}-\mathbf{k}')+1\to 2] \tag{4.3}$$

$$where\ \ \Gamma(1,\omega,\mathbf{k}) = P(1,\omega,k_z)\Big(R(1,k_z)S(k_xL) - 2e^{-ik_xL_1}\Big) \tag{4.4}$$

For a Maxwellian $f_0(v)$, we have

$$\frac{\mathbf{k}'\cdot\frac{df_0}{d\mathbf{v}}}{\omega'-\mathbf{k}'\cdot\mathbf{v}} == -\frac{m}{T}f_0(-1+\frac{\omega'}{\omega'-\mathbf{k}'\cdot\mathbf{v}}) \tag{4.5}$$

where –1 can be assumed to contribute nothing to the velocity integral due to phase mixing. Thus $\mathbf{k}' \cdot d/d\mathbf{v}$ in Eq. (4.3) will be replaced by $-m\omega'/T$. Equation (4.3) will be used for investigation of echo occurrence.

The various cross terms in the product $\Gamma(1)\Gamma(2)$ are the candidates of echo resonances. We choose to investigate a cross term which is 1–term in $\Gamma(1)$ multiplied by the exponential term in $\Gamma(2)$ in Eq. (4.3):

$$P(1,\omega',k_z')R(1,k_z')S(k_x'L)P(2,\omega-\omega',k_z-k_z')(-2)e^{-i(k_x-k_x')L_2} \tag{4.6}$$

With the above term, the t–inversion $\int d\omega/2\pi\ e^{-i\omega t}(\cdots)$ can be easily carried out by simply putting $\omega' \to -\omega_1$ and $\omega \to -\omega_3 = -(\omega_1+\omega_2)$:

$$\mathbf{E}^{(2)}(t,\mathbf{k}) = C_1\ e^{i\omega_3 t}\ \int d^2v f_0\ \frac{\mathbf{k}}{k^2}\frac{1}{(\omega_3+\mathbf{k}\cdot\mathbf{v})^2}\ e^{-ik_z\zeta_2}\ e^{-ik_xL_2}$$

$$\times\ \int dk_z' e^{i(\zeta_2-\zeta_1)k_z'}R(1,k_z')\int dk_x'\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{(\mathbf{k}-\mathbf{k}')^2k'^2(\omega_1+\mathbf{k}'\cdot\mathbf{v})}\ S(k_x'L)e^{ik_x'L_2} \tag{4.7}$$

where C_1 is a constant consisting of unessential factors, and we have taken the dielectric functions out of the integral. We eventually can show that they become the product $[\varepsilon(-\omega_1)\varepsilon(-\omega_2)\varepsilon(-\omega_3)]^{-1}$ with the $\mathbf{k}$–dependance obliterated.

Let us write explicitly the inversion integral of Eq. (4.7) with respect to $\mathbf{k}$:

$$\mathbf{E}^{(2)}(t,x,z) = C_2\ e^{i\omega_3 t}\ \int \frac{d^2vf_0}{v_x^2}\ \int dk_z e^{ik_z(z-\zeta_2)}\int dk_x\ e^{ik_x(x-L_2)}\frac{\mathbf{k}}{k^2}\frac{1}{(k_x+\frac{k_zv_z+\omega_3}{v_x})^2}$$

$$\times\ \int dk_z' e^{i(\zeta_2-\zeta_1)k_z'}R(1,k_z')\int dk_x'\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{(\mathbf{k}-\mathbf{k}')^2k'^2(\omega_1+\mathbf{k}'\cdot\mathbf{v})}\ S(k_x'L)e^{ik_x'L_2} \tag{4.8}$$

The main contribution to the $\int dk_x$–integral comes from the double pole at

$$k_x = -\frac{k_zv_z+\omega_3}{v_x} \tag{4.9}$$

The residue is obtained by taking $\partial/\partial k_x$ [Integrand$\times(k_x+\frac{k_zv_z+\omega_3}{v_x})^2$] and substituting Eq. (4.9) for k_x. Here it is sufficient to differentiate the exponential functions only because they yield asymptotically dominant result. After performing the $\int dk_x$–integration, we have

$$\mathbf{E}^{(2)}(t,x,z) = C_3(x-L_2)\ e^{i\omega_3 t}\ \int \frac{d^2vf_0}{v_x^2}\ exp\left[-i(x-L_2)\omega_3/v_x\right]$$

$$\times\ \int dk_z\ \frac{\mathbf{k}}{k^2}\ e^{ik_z(z-\zeta_2)}H(-Im\ \frac{k_zv_z}{v_x})\ exp[-i(x-L_2)k_zv_z/v_x]$$

$$\int dk_z' e^{i(\zeta_2-\zeta_1)k_z'}R(1,k_z')\int dk_x'\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{(\mathbf{k}-\mathbf{k}')^2k'^2(\omega_1+\mathbf{k}'\cdot\mathbf{v})}\ S(k_x'L)e^{ik_x'L_2} \tag{4.10}$$

$$where\ \ \mathbf{k} = \left(-\frac{k_zv_z+\omega_3}{v_x},k_z\right) \tag{4.11}$$

Integral $\int dk'_x$ can be carried out by picking up the simple pole at $\omega_1 + \mathbf{k}' \cdot \mathbf{v} = 0$ or at $k'_x = -(k'_z v_z + \omega_1)/v_x$, giving

$$\mathbf{E}^{(2)}(t,x,z) = C_4\ (x - L_2)\ e^{i\omega_3 t} \int \frac{d^2 v f_0}{v_x^3}\ e^{-i(x-L_2)\omega_3/v_x}\ e^{-iL_2\omega_1/v_x}$$

$$\times \int dk_z \frac{\mathbf{k}}{k^2} H(-Im\ \frac{k_z v_z}{v_x})\ exp[ik_z(z-\zeta_2) - i(x-L_2)k_z v_z/v_x]$$

$$\times \int dk'_z H(-Im\ \frac{k'_z v_z}{v_x}) exp\ [i(\zeta_2 - \zeta_1)k'_z - iL_2 k'_z v_z/v_x] R(1,k'_z) \frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{(\mathbf{k}-\mathbf{k}')^2 k'^2}\ S(k'_x L) \tag{4.12}$$

$$where\ \ \mathbf{k}' = \Big(-\frac{k'_z v_z + \omega_1}{v_x}, k'_z\Big) \tag{4.13}$$

In carrying out the above integral, we pick up the simple poles at $k^2 = 0$ and $k'^2 = 0$. Equations (4.11) and (4.13) give respectively

$$k^2 = \frac{v^2}{v_x^2}(k_z - k_{z+})(k_z - k_{z-}) \tag{4.14}$$

$$k'^2 = \frac{v^2}{v_x^2}(k'_z - k'_{z+})(k'_z - k'_{z-}) \tag{4.15}$$

where

$$k_{z\pm} = -\frac{\omega_3}{v^2}(v_z \pm i v_x) \tag{4.16}$$

$$k'_{z\pm} = -\frac{\omega_1}{v^2}(v_z \pm i v_x) \tag{4.17}$$

Which simple pole we should take depends upon the contours that must be used. In any case, we have $\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{(\mathbf{k}-\mathbf{k}')^2} = \frac{1}{2}$. Thus integral $\int dk'_z$ in Eq. (4.12) is written as

$$\int dk'_z(\cdots) = \frac{iv_x}{4\omega_1}\int dk'_z \Big(\frac{1}{k'_z - k'_{z+}} - \frac{1}{k'_z - k'_{z-}}\Big) exp\ [i(\zeta_2 - \zeta_1)k'_z - iL_2 k'_z v_z/v_x]$$

$$\times\ H(-Im\ \frac{k'_z v_z}{v_x})\ R(1,k'_z) S(-\frac{\omega_1 + v_z k'_z L}{v_x} L) \tag{4.18}$$

Let us assume

$$\theta \equiv \zeta_2 - \zeta_1 - L_2 \frac{v_z}{v_x} < 0 \tag{4.19}$$

Then the k'_z-contour should encircle the lower-half k'_z plane. Thus we obtain

$$\int dk'_z(\cdots) = \frac{\pi v_x}{2\omega_1}\Big[H(v_x) e^{i\theta k'_{z+}} R(1,k'_{z+}) S(-ik'_{z+}L)$$

$$-H(-v_x) e^{i\theta k'_{z-}} R(1,k'_{z-}) S(ik'_{z-}L)\Big] \tag{4.20}$$

Let us assume

$$\theta_0 \equiv z - \zeta_2 - (x - L_2)\frac{v_z}{v_x} < 0 \tag{4.21}$$

Then the k_z-contour in Eq. (4.12) should encircle the lower-half k_z plane. Thus we obtain

$$\int dk_z(\cdots) = \frac{iv_x}{2\omega_3}\int dk_z\ \mathbf{k}\left(\frac{1}{k_z - k_{z+}} - \frac{1}{k_z - k_{z-}}\right)exp\ [ik_z\theta_0]H(-Im\ \frac{k_z v_z}{v_x})$$

$$= \frac{\pi v_x}{\omega_3}\left[\mathbf{k}(k_{z+})H(v_x)e^{ik_{z+}\theta_0} - \mathbf{k}(k_{z-})H(-v_x)e^{ik_{z-}\theta_0}\right] \tag{4.22}$$

$$\int dk'_z(\cdots)\int dk_z(\cdots) = \frac{\pi^2 v_x^2}{2\omega_1\omega_3}\Big[H(v_x)\mathbf{k}(k_{z+})exp[i\theta k'_{z+} + i\theta_0 k_{z+}]R(1, k'_{z+})S(-ik'_{z+}L) +$$

$$+H(-v_x)\mathbf{k}(k_{z-})exp[i\theta k'_{z-} + i\theta_0 k_{z-}]R(1, k'_{z-})S(ik'_{z-}L)\Big] \tag{4.23}$$

The series $S(\phi)$, where ϕ represents either $-ik'_{z+}L$ or $ik'_{z-}L$, can be decomposed:

$$S(\phi) = \frac{1}{2} + cos\phi + cos2\phi + cos3\phi + \cdots$$

$$= \frac{1}{2}[\frac{1}{2} + e^{i\phi} + e^{2i\phi} + e^{3i\phi} + \cdots + \frac{1}{2} + e^{-i\phi} + e^{-2i\phi} + e^{-3i\phi} + \cdots] \tag{4.24}$$

We shall use the dispersion relation in Eq. (3.30) to evaluate the above series. Eq. (3.30) for the symmetric mode can be written as

$$\frac{i}{2\pi\varepsilon}\int_{-\infty}^{\infty} d\xi(\frac{1}{\xi + ik_z} - \frac{1}{\xi - ik_z})[\ \frac{1}{2} + e^{i\xi L} + e^{2i\xi L} + e^{3i\xi L} + \cdots] = -\frac{1}{2} \tag{4.25}$$

This can be contour-integrated by surrounding the upper ξ-plane. We have:

If Re $k_z < 0$, $\frac{1}{2} + e^{k_z L} + e^{2k_z L} + e^{3k_z L} + \cdots = \frac{\varepsilon}{2}$. If Re $k_z > 0$, $\frac{1}{2} + e^{-k_z L} + e^{-2k_z L} + e^{-3k_z L} + \cdots = -\frac{\varepsilon}{2}$. If this result is applied to Eq. (4.24), we can write

$$S(\phi) = \frac{\varepsilon}{4}H(Re[-i\phi]) + X_+ H(Re[i\phi]) - \frac{\varepsilon}{4}H(Re[i\phi]) + X_- H(Re[-i\phi]) \tag{4.26}$$

where $X_\pm$ are unknown quantities which will be discarded in the following by assuming that they contribute only to phase-mixing integrals. Thus, we write

$$S(ik'_{z-}L) = \frac{\varepsilon}{4}H(-v_z) - \frac{\varepsilon}{4}H(v_z) \tag{4.27}$$

$$S(-ik'_{z+}L) = \frac{\varepsilon}{4}H(v_z) - \frac{\varepsilon}{4}H(-v_z) \tag{4.28}$$

Equation (3.45) and the above equations yield

$$R(k'_{z-})S(ik'_{z-}L) = \frac{1+\varepsilon}{4}H(v_z)(e^{k'_{z-}(L-L_1)} + e^{L_1 k'_{z-}}) + \frac{1-\varepsilon}{4}H(-v_z)(e^{-k'_{z-}(L-L_1)} + e^{-L_1 k'_{z-}})$$

$$R(k'_{z+})S(-ik'_{z+}L) = -\frac{1+\varepsilon}{4}H(v_z)(e^{k'_{z+}(L-L_1)} + e^{L_1 k'_{z+}}) - \frac{1-\varepsilon}{4}H(-v_z)(e^{-k'_{z+}(L-L_1)} + e^{-L_1 k'_{z+}})$$

Therefore the velocity integral in Eq. (4.12) consists of the following four parts.

$$\int_0^\infty dv_x \int_0^\infty dv_z(\cdots)(e^{\varphi_1} + e^{\varphi_1'}) : \tag{4.29}$$

$$\varphi_1 = k'_{z+}(L - L_1) + i\theta k'_{z+} + i\theta_0 k_{z+} - i(x - L_2)\omega_3/v_x - iL_2\omega_1/v_x \tag{4.30}$$

$$\varphi_1' = k'_{z+}L_1 + i\theta k'_{z+} + i\theta_0 k_{z+} - i(x - L_2)\omega_3/v_x - iL_2\omega_1/v_x \tag{4.31}$$

$$\int_0^\infty dv_x \int_{-\infty}^0 dv_z(\cdots)(e^{\varphi_2} + e^{\varphi_2'}) : \tag{4.32}$$

$$\varphi_2 = -k'_{z-}(L - L_1) + i\theta k'_{z+} + i\theta_0 k_{z+} - i(x - L_2)\omega_3/v_x - iL_2\omega_1/v_x \tag{4.33}$$

$$\varphi_2' = -k'_{z-}L_1 + i\theta k'_{z+} + i\theta_0 k_{z+} - i(x - L_2)\omega_3/v_x - iL_2\omega_1/v_x \tag{4.34}$$

$$\int_{-\infty}^0 dv_x \int_0^\infty dv_z(\cdots)(e^{\varphi_3} + e^{\varphi_3'}) : \tag{4.35}$$

$$\varphi_3 = k'_{z-}(L - L_1) + i\theta k'_{z-} + i\theta_0 k_{z-} - i(x - L_2)\omega_3/v_x - iL_2\omega_1/v_x \tag{4.36}$$

$$\varphi_3' = k'_{z-}L_1 + i\theta k'_{z-} + i\theta_0 k_{z-} - i(x - L_2)\omega_3/v_x - iL_2\omega_1/v_x \tag{4.37}$$

$$\int_{-\infty}^0 dv_x \int_{-\infty}^0 dv_z(\cdots)(e^{\varphi_4} + e^{\varphi_4'}) : \tag{4.38}$$

$$\varphi_4 = -k'_{z-}(L - L_1) + i\theta k'_{z-} + i\theta_0 k_{z-} - i(x - L_2)\omega_3/v_x - iL_2\omega_1/v_x \tag{4.39}$$

$$\varphi_4' = -k'_{z-}L_1 + i\theta k'_{z-} + i\theta_0 k_{z-} - i(x - L_2)\omega_3/v_x - iL_2\omega_1/v_x \tag{4.40}$$

We are interested in the imaginary parts of the phases. After some algebra we obtain

$$Im\ \varphi_1 = \frac{v_x}{v^2}[\omega_2 L_2 - \omega_1(L - L_1) - x\omega_3] - \frac{v_z}{v^2}[\omega_3(z - \zeta_2) + \omega_1(\zeta_2 - \zeta_1)] \tag{4.41}$$

$$Im\ \varphi_1' = -\frac{v_x}{v^2}[\omega_3(x - L_2) + \omega_1(L_1 + L_2)] - \frac{v_z}{v^2}[\omega_3(z - \zeta_2) + \omega_1(\zeta_2 - \zeta_1)] \tag{4.42}$$

$$Im\ \varphi_2 = -\frac{v_x}{v^2}[\omega_1(L - L_1) + \omega_1 L_2 + (x - L_2)\omega_3] - \frac{v_z}{v^2}[\omega_3(z - \zeta_2) + \omega_1(\zeta_2 - \zeta_1)] \tag{4.43}$$

$$Im\ \varphi_2' = -\frac{v_x}{v^2}[\omega_1(L_1 + L_2) + (x - L_2)\omega_3] - \frac{v_z}{v^2}[\omega_3(z - \zeta_2) + \omega_1(\zeta_2 - \zeta_1)] \tag{4.44}$$

$$Im\ \varphi_3 = -\frac{v_x}{v^2}[\omega_1(L_1 - L) + \omega_1 L_2 + (x - L_2)\omega_3] - \frac{v_z}{v^2}[\omega_3(z - \zeta_2) + \omega_1(\zeta_2 - \zeta_1)] \tag{4.45}$$

$$Im\ \varphi_3' = -\frac{v_x}{v^2}[-\omega_1 L_1 + \omega_1 L_2 + (x - L_2)\omega_3] - \frac{v_z}{v^2}[\omega_3(z - \zeta_2) + \omega_1(\zeta_2 - \zeta_1)] \tag{4.46}$$

$$Im\ \varphi_4 = -\frac{v_x}{v^2}[\omega_1(L-L_1)+\omega_1 L_2+(x-L_2)\omega_3] - \frac{v_z}{v^2}[\omega_3(z-\zeta_2)+\omega_1(\zeta_2-\zeta_1)] \tag{4.47}$$

$$Im\ \varphi_4' = -\frac{v_x}{v^2}[\omega_1(L_1+L_2)+(x-L_2)\omega_3] - \frac{v_z}{v^2}[\omega_3(z-\zeta_2)+\omega_1(\zeta_2-\zeta_1)] \tag{4.48}$$

Among the above integrals, φ_3 and φ_3' yield acceptable echo coordinates. Putting $Im\ \varphi_3 = 0$ gives an echo spot:

$$x_{echo} = \frac{\omega_1(L-L_1)+\omega_2 L_2}{\omega_3},\ z_{echo} = \frac{\omega_1\zeta_1+\omega_2\zeta_2}{\omega_3} \tag{4.49}$$

Putting $Im\ \varphi_3' = 0$ gives an echo spot:

$$x_{echo} = \frac{\omega_1 L_1+\omega_2 L_2}{\omega_3},\ z_{echo} = \frac{\omega_1\zeta_1+\omega_2\zeta_2}{\omega_3} \tag{4.50}$$

Note that z_{echo}'s are the same for the above phases.

We obtained two echo spots as given by Eqs. (4.49)–(4.50). Among these, x_{echo} in Eq. (4.49), which depends on the slab thickness L, is characteristic of slab. It appears that the other can be in a semi-bounded or an infinite plasma as well. The echo spots that we obtained need to be checked against the inequalities $\theta_0 < 0$ and $\theta < 0$ postulated at the outset of contour integration. We have a great variety of combinations of the parameters $\omega_1, \omega_2, L_1, L_2, \zeta_1,$ and ζ_2. It appears that there is no difficulty to have the premises $\theta_0 < 0$ and $\theta < 0$, and the condition $0 < x_{echo} < L$ satisfied.

5 DISCUSSION

Equations (4.49) and (4.50) are the main results of this work in locating the echo spots associate with the the surface of the symmetric mode in a slab plasma launched by two oscillating charges at $(x, z) = (L_1, \zeta_1)$ and (L_2, ζ_2). The echo at Eq. (4.49) is slab surface wave-proper. Our search for the echo spots are not exhaustive; the echoes associated with the anti-symmetric mode should be investigated separately, and we put aside many other product terms in Eq. (4.3). It appears that we have diversity of echo locations in a slab plasma. It was reported of experiment that we have multiple echo spots in a semi-bounded plasma [11]. The diversity appears to be due to reflections of the wave at the interface. Mathematically, the discontinuity of $E_x(x)$ at $x = \pm nL$ is responsible for the diversity.

In reality, bounded plasmas are usual rather than exceptional. Important literatures to get acquainted with this field are References [10] and [13], among others. Earlier authors investigated in semi-bounded plasmas the surface wave echoes [4,5,8] and the nonlinear wave interactions [14]. The investigation of the surface waves in a slab geometry is scarcely found in the literature. The experimental report appears to be mainly with regard to a semi-bounded plasma. The slab plasma appears to be worthy of more attention. The kinetic theory of the surface waves in a slab has been dealt with in References [9] and [15]. In Reference [15], the electric field is expanded in a cosine series. In Reference [9], which is the basis of the present work, automatically derives the exponential series by extending the electric field in an oddly manner.

6 CONCLUSION

In a plasma slab, the spatial echoes can occur at various spots. The diversity of echo spots is firstly due to the multiplicity of the wave modes, the symmetric and the anti-symmetric modes of the surface wave, each satisfying different dispersion relation, and secondly due to the reflections of the waves off the two interfaces. In this work, we investigated the spatial echoes associated with the symmetric mode. Our investigation is not exhaustive since an exemplary quadratic interaction term was chosen. The mathematical analysis accompanying the relevant contour integration shown in this work would be useful for calculation of different interaction terms. The temporal echoes in a slab geometry would be an interesting subject of future investigation.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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