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On Varanovskaya Type Theorem for Generalized Bernstein-Chlodowsky Polynomials

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper we proved Varanovskaya type theorem for generalized Bernstein-Chlodowsky polynomials.

Keywords: Generalized Bernstein-Chlodowsky operator; approximation theorem; analog of Bernstein type theorem.

1 Introduction

The polynomial constructed by Bernstein in 1912 for a continuous function has the form

$$\overline{B_n}(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k (x)^k (1-x)^{n-k},\ 0\le x\le 1,\ k=0,1,...n.$$

In 1932, Bernstein's follower Chlodowsky had constructed the increasing sequences of polynomials

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$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k},\ 0\le x\le b_n, \tag{1.1}$$

that was named as Bernstein-Chlodowsky polynomials some time later,

where

$$\lim_{n\to\infty} b_n=\infty,\quad \lim_{n\to\infty}\frac{b_n}{n}=0.$$

Note that, the condition imposed on b_n in the definition of classic Bernstein-Chlodowsky operators does not provide continuous convergence of $B_n(f;x)$ polynomials to the function $f(x)$.

Note here the papers [1-5,6,7,8] related directly to the Bernstein-Chlodowsky polynomials and their generalizations and the paper [4,9,10] generalizing the results of the paper [11] on infinitely increasing interval from which we can also conclude a number of theorems on these polynomials. Some main properties of the polynomials (1.1) were stated in the monograph [12]. About applications of general positive linear operator we refer to [13,14].

In the paper we prove an approximation theorem for Bernstein-Chlodowsky polynomial.

2 Preliminaries and Auxiliary Results

We have

$$B_n(f;x)-x^2=\frac{x(b_n-x)}{n} \text{ for the function } f(t)=t^2.$$

Therefore,

$$\sup_{x\in[0.b_n]}\left|B_n(t^2;x)-x^2\right|=\frac{b_n^2}{4n}.$$

Then for convergence to zero as $n\to\infty$ of the right hand part of this equality does not suffice the fullfillment of condition $\frac{b_n}{n}\to\infty$.

We'll impose the stronger condition on b_n

$$\lim_{n\to\infty}\frac{b_n}{\sqrt[4]{n}}=0\ .$$

Hence it follows that for any natural number r

$$\lim_{n\to\infty}\frac{b_n^r}{n^{\frac{r}{4}}}=0.$$

Lemma 1.1. *For Bernstein-Chlodowsky polynomial (1.1) the following properties*

$$B_n(1;x)=1$$
$$B_n(t;x)=x$$
$$B_n(t^2;x)=x^2+\frac{x(b_n-x)}{n}$$

hold.

Theorem 2.1. *Let the function* $f'(x)$ *be uniformly continuous on the semiaxis and* $B_n(f;x)$ *be Bernstein-Chlodowsky polynomials of order* n *for the function* $f(x)$. *Then for any* $x\in[0,b_n]$

$$|B_n(f;x)-f(x)|\le\frac{3}{2}\omega\left(f;\frac{b_n}{\sqrt{n}}\right).$$

Remark 1.1. *Note that theorem 2.1 was proved in [1].*

3 Main Results

Now we reduce the main results of this paper.

Theorem 3.1. *Let* $f\in C^2[0,\infty)$ *and* $B_n(f,x)$ *be the Bernstein-Chlodowsky polinomials for function* f. *Let a sequence* $\{b_n\}$ *satisfy condition*

$$\lim_{n\to\infty}M_n(f')\frac{b_n^2}{n}=0.$$

Then for every $x\in[0,b_n]$ *the asymptotical equality*

$$B_n(f;x)=f(x)+\frac{f''(x)}{2n}x(b_n-x)+\rho_n\,,\ \ \rho_n\to 0,n\to\infty \tag{3.1}$$

Proof. We have

$$f\left(\frac{kb_n}{n}\right)=f(x)+f'(x)\left(\frac{kb_n}{n}-x\right)+\left[\frac{f''(x)}{2}+\lambda\left(\frac{kb_n}{n}\right)\right]\left(\frac{kb_n}{n}-x\right)^2. \tag{3.2}$$

By (3.1), we have

$$B_n(f;x)=\sum_{k=0}^{n}\left\{f(x)+f'(x)\left(\frac{kb_n}{n}-x\right)+\right.$$

$$\left.+\left[\frac{f''(x)}{2}+\lambda\left(\frac{kb_n}{n}\right)\right]\left(\frac{kb_n}{n}-x\right)^2\right\}\times C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}=$$

$$=\sum_{k=0}^{n}f(x)\,C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}+$$

$$+\sum_{k=0}^{n}f'(x)\left(\frac{kb_n}{n}-x\right)C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}+$$

$$+\sum_{k=0}^{n}\frac{f''(x)}{2}\left(\frac{kb_n}{n}-x\right)^2C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}+$$

$$+\sum_{k=0}^{n}\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}=$$

$$=f(x)+f'(x)(x-x)+\frac{f''(x)}{2}\left(x^2+\frac{x(b_n-x)}{n}-2x^2+x^2\right)+$$

$$+\sum_{k=0}^{n}\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}=$$

$$=f(x)+\frac{f''(x)}{2n}x(b_n-x)+\sum_{k=0}^{n}\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}.$$

$$B_n(f;x)=f(x)+\frac{f''(x)}{2n}x(b_n-x)+r_n,$$

where

$$r_n=\sum_{k=0}^{n}\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}.$$

We take arbitrary $\varepsilon>0$ and choosing sufficient large $n\in\mathrm{N}$ such that

$$\left|\frac{kb_n}{n}-x\right|<\frac{b_n}{\sqrt[4]{n}}\ .$$

Then $\left|\lambda\left(\frac{kb_n}{n}\right)\right| < \frac{\varepsilon}{2}$. We divide r_n as the sum of two terms:

$$r_n = \sum_{\left|\frac{kb_n}{n}-x\right|<\frac{b_n}{\sqrt[4]{n}}} \lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k} +$$

$$+\sum_{\left|\frac{kb_n}{n}-x\right|\geq\frac{b_n}{\sqrt[4]{n}}} \lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}.$$

Therefore

$$\left|\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2\right| \leq$$

$$\leq M_f\left(1+\left(\frac{kb_n}{n}\right)^m+\left(1+x^m\right)+\left(1+x^m\right)\left(\frac{kb_n}{n}-x\right)+\left(1+x^m\right)\left(\frac{kb_n}{n}-x\right)^2\right) \leq$$

$$\leq M_f b_n^m\left(1+\left|\frac{kb_n}{n}-x\right|\frac{\left|\frac{kb_n}{n}-x\right|}{b_n}+\left(\frac{kb_n}{n}-x\right)^2\right) \leq$$

$$\leq M_f b_n^m\left(\frac{\left(\frac{kb_n}{n}-x\right)^2}{\delta^2}+\frac{\left(\frac{kb_n}{n}-x\right)^2}{\delta}+\left(\frac{kb_n}{n}-x\right)^2\right) \leq$$

$$\leq M_f b_n^m\frac{\left(\frac{kb_n}{n}-x\right)^2}{\delta^2}.$$

Thus

$$r_n \leq \sum_{\left|\frac{kb_n}{n}-x\right|<\frac{b_n}{\sqrt[4]{n}}} \frac{\varepsilon}{2}\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k} +$$

$$+Mb_n^m\sum_{\left|\frac{kb_n}{n}-x\right|\geq\frac{b_n}{4\sqrt{n}}} \frac{\left(\frac{kb_n}{n}-x\right)^{2m}}{\left(\frac{b_n}{\sqrt[4]{n}}\right)} C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k} \leq$$

$$\leq \frac{\varepsilon}{2}\frac{b_n^2}{\sqrt{n}} + Mb_n^m \frac{n^{\frac{m}{2}}}{b_n^{2m}} \sum_{k=0}^{n} \left(\frac{kb_n}{x} - 1\right)^{2m} C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} \leq$$

$$\leq \frac{\varepsilon}{2}\frac{b_n^2}{\sqrt{n}} + Mb_n^m \frac{n^{\frac{m}{2}}}{b_n^{2m}} \frac{b_n^{2m}}{n^{2m}} \sum_{k=0}^{n} \left(k - \frac{nx}{b_n}\right)^{2m} C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}.$$

Taking into account (3.1) in the last inequality, we get

$$r_n \leq \frac{\varepsilon}{2}\frac{b_n^2}{\sqrt{n}} + M\frac{b_n^m}{n^{\frac{3m}{2}}} K(2m) n^m = \frac{\varepsilon}{2}\frac{b_n^2}{\sqrt{n}} + \frac{b_n^m}{n^{\frac{m}{2}}} MK(2m).$$

For $m = 4$, we have

$$r_n \leq \frac{\varepsilon}{2}\frac{b_n}{\sqrt{n}} + \frac{b_n^4}{n^2} MK(8).$$

Since $\frac{b_n^2 MK(8)}{n^{\frac{3}{2}}} < \frac{\varepsilon}{2}$, then

$$r_n < \frac{b_n}{\sqrt{n}}\varepsilon.$$

We denote $\rho_n = \frac{\sqrt{n}}{b_n^2} r_n$.

This completes the proof of theorem.

4 Conclusion

Thus, we proved Varanovskaya type theorem for generalized Bernstein-Chlodowsky polynomials.

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Competing Interests

Author has declared that no competing interests exist.

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